SUBMARINE CONTROL AND DYNAMIC TRIM ESTIMATION

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by

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ABSTRACT

Optimal estimation techniques are manipulated to demonstrate the feasibility of using a Wiener filter for trim analysis of a submarine. The essence of the problem is the estimation of deviations of the submarine's weight and longitudinal center of gravity from those values which enable the submarine to maintain ordered depth and angle of pitch with minimal control surface deflections.

Linearized equations of motion are developed and put into state vector notation. The state vector itself contains the variables which are to be estimated. Optimal control techniques are used to design a linear regulator which is used to control the simulated submarine in the vertical plane. Two types of filters are designed. The first is sub-optimal in that its gains are derived by pole placement techniques which are used to ensure filter stability. The second filter is a Wiener filter, in that its gains come from the steady state solution of the matrix Riccati equation. A least squares smoother is designed to smooth filtered estimates of the variables of interest.

Simulations on a digital computer are used to demonstrate the ability of the filters to estimate the submarine's state of trim in the presence of disturbances to the submarine and the measurement instruments. It is also demonstrated that the Wiener filter is capable of estimating the severity of a flooding casualty on the submarine.

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Nomenclature

a_i	Non-dimensional value of A_i $(A_i / 1/2 \rho \ell^2)$
a_{ij}	Element from the ith row and jth column of $\underline{A}$.
$\underline{A}$	Coefficient matrix of the state vector $(\underline{x})$.
$\underline{A}_c$	Coefficient matrix for the reduced state vector of the controller
$\underline{A}_f$	Coefficient matrix for the reduced state vector of the filter
A_i	Constant used in the representation of propeller thrust in the surge equation
b_i	Non-dimensional value of $B_i (B_i / 1/2 \rho \ell^2)$
b_{ij}	Element from the ith row and jth column of $\underline{B}$
b'_{ij}	Element from the ith row and jth column of $\underline{B}'$
$\underline{B}$	Coefficient matrix of the input vector $(\underline{u})$
$\underline{B}'$	Coefficient matrix of the external disturbance vector $(\underline{F})$
$\underline{B}_c$	Coefficient matrix of the control vector $(\underline{u}_c)$
$\underline{B}_f$	Coefficient matrix of the filter input vector $(\underline{u}_f)$
B_i	Coefficient used in the representation of propeller thrust in the surge equation
c_i	Non-dimensional value of $C_i (C_i / 1/2 \rho \ell^2)$

$\underline{C}$	Coefficient matrix which operates on a state vector ($\underline{x}$) to yield an output vector ($\underline{y}$)
$\underline{C}_c$	Coefficient matrix which reduces the measurement signal of the state vector ($\underline{z}$) to the measurement signal of the reduced controller state vector ($\underline{z}$) to the measurement signal of the reduced controller state vector ($\underline{z}_c$).
$\underline{C}_f$	Coefficient matrix which transforms the reduced filter state vector ($\underline{x}_f$) vector of the filter
$\underline{C}'_f$	Coefficient matrix which reduces the measurement signal of the state vector ($\underline{z}$) to the measurement signal of the reduced filter state vector ($\underline{z}_f$)
C_i	Coefficient used in the representation of propeller thrust in the surge equation
$\underline{C}_{uf}$	Coefficient matrix which extracts elements of the input vector for the filter ($\underline{u}_f$) from the measurement signal of the state vector ($\underline{z}$)
$\underline{C}_{\delta c}$	Coefficient matrix which extracts the planes deflection vector ($\underline{\delta}$) from the state vector ($\underline{x}$)
$C_{\eta 1}$	Constant used in the representation of the steady state value of (η - 1) in the linearized EOM
$C_{\eta 2}$	First order coefficient used in the representation of (η - 1) as a function of Δu in the linearized EOM
CB	Center of buoyancy of the submarine
CG	Center of mass of the submarine
EOM	Equations of motion
E( )	Symbol used to denote the expected value or mean value of the variable within the parentheses

$\underline{F}$	Resultant external force on the submarine or steady state and external disturbance vector for the linearized EOM
$F_{Zext}, F_{Mext}, F_{Xext}$	External disturbances on the submarine in heave, pitch and surge respectively
F_{Zs}, F_{Ms}, F_{Xs}	Resultant stady state forces and moment (excluding external distrubances) for the linearized EOM in heave, pitch, and surge respectively
g	Acceleration due to gravity
$\underline{H}_G$	Angular momentum of the submarine
$\hat{i}$	Unit vector in the x direction
$\underline{I}_G$	Inertia matrix of the submarine about its CG
I_{Gy}	Moment of inertia of the submarine about an axis parallel to the y axis and through the CG
I_{yo}	Equilibrium value of the submarine's moment of inertia for the linearized EOM
$\hat{j}$	Unit vector in the y direction
$\hat{k}$	Unit vector in the z direction
k_{ij}	Element from the ith row and jth column of $\underline{K}_c$
$\underline{K}$	Feedback gains matrix
$\underline{K}_c$	Controller feedback gains matrix
ℓ	Length of the submarine
LCG	Longitudinal CG of the submarine (x_G)

m	Mass of submarine, including water in free-flooding spaces
m_o	Mass of the submarine, including free flooding water, when "in trim"
M	Hydrodynamic moment component about y axis (pitching moment)
M_*	Pitching moment when body angle (α) and control surface angles are zero
M_q	First order coefficient used in representing M as a function of q
$M_{q\eta}$	First order coefficient used in representing M_q as a function of $(\eta - 1)$
$M_{\dot{q}}$	Coefficient used in representing M as a function of $\dot{q}$
$M_{q\|q\|}$	Second order coefficient used in representing M as a function of q
$M_{\|q\|\delta_s}$	Coefficient used in representing M_{δ_s} as a function q
M_w	First order coefficient used in representing M as a function of w
$M_{w\eta}$	First order coefficient used in representing M_w as a function of $(\eta - 1)$
$M_{\dot{w}}$	Coefficient used in representing M as a function of $\dot{w}$
$M_{\|w\|}$	First order coefficient used in representing M as a function of w; equal to zero for symmetrical function
$M_{\|w\|q}$	Coefficient used in representing M_q as a function of w

$M_{w\|w\|}$	Second order coefficient used in representing M as a function of w
$M_{w\|w\|\eta}$	First order coefficient used in representing $M_{w\|w\|}$ as a function of $(\eta - 1)$
M_{ww}	Second order coefficient used in representing M as a function of w; equal to zero for symmetrical function
$M_{\delta b}$	First order coefficient used in representing M as a function of δ_b
$M_{\delta s}$	First order coefficient used in representing M as a function of δ_s
$M_{\delta s \eta}$	First order coefficient used in representing $M_{\delta s}$ as a function of $(\eta - 1)$
$M_{\Delta mg}$	First order coefficient used in the representation of M_t as a function of W_e in the linearized EOM
$M_{\Delta \dot{m}g}$	First order coefficient used in the representation of M_t as a function of $\dot{W}$ in the linearized EOM
$M_{\Delta u}$	First order coefficient used in the representation of M as a function of Δu in the linearized EOM
$M_{\Delta \dot{u}}$	First order coefficient used in the representation of M as a function of $\dot{u}$ in the linearized EOM
$M_{\Delta w}$	First order coefficient used in the representation of M as a function of Δw in the linearized EOM
$M_{\Delta \dot{w}}$	First order coefficient used in the representation of M as a function of $\dot{w}$ in the linearized EOM
$M_{\Delta \Theta e}$	First order coefficient used in the representation of M_t as a function of Θ_e in the linearized EOM

$M_{\Delta\dot{\theta}e}$	First order coefficient used in the representation of M as a function of $\dot{\theta}_e$ in the linearized EOM
$M_{\Delta\delta s}$	First order coefficient used in the representation of M as a function of δ_s in the linearized EOM
$\underline{P}$	Weighting matrix in the quadratic performance index
$\underline{P}_c$	Control effort weighting matrix
$\underline{P}_f$	Measurement noise covariance matrix

q	Angluar velocity component of body axes reltive to fluid
$\underline{Q}$	Weighting matrix in the quadratic performance index
$\underline{Q}_c$	Weighting matrix of the state vector
$\underline{Q}_f$	External noise covariance matrix
$\underline{R}_G$	Location of CG relative to body axes
$\underline{R}$	Solution to matrix Riccati equation
$\underline{T}$	Resultant torque about the body axes
$\underline{T}_G$	Resultant torque about the submarine's CG
u	Longitudinal component of the velocity of the body axes relative to the fluid
Δu	Deviation of u from u_o in linearized EOM
$\underline{u}$	Input vector
u_c	Command speed: steady state value of ahead speed for a given propeller rpm when body angle and control surface angles are zero

$\underline{u}_c$	Unconstrained optimal control vector
$\underline{u}'_c$	Constrained control effort vector
$\underline{u}_f$	Filter input vector
u_o	Longitudinal component of the equilibrium velocity of the body axes relative to the fluid
u_{rel}	Longitudinal component of the velocity of the CG relative to the body axes
$\underline{U}$	Velocity of the body axes relative to the fluid
$\underline{U}_G$	Velocity of the submarine's CG relative to the inertial axes
U_o	Equilibrium speed of the body axes relative to the fluid
$\underline{U}_{rel}$	Velocity of the CG relative to the body axes
$\underline{v}$	Measurement noise vector
$\underline{V}$	Coefficient matrix of the left hand side of the EOM
$\underline{V}'$	Eigenvector matrix from Potter's method of solving the reduced Riccati equation
w	Component of $\underline{U}$ in the direction of the z axis
$\underline{w}$	External distrubance vector
w_o	Equilibrium value of w for the linearized EOM
w_{rel}	Component of $\underline{U}_{rel}$ in the direction of the z axis

W	Weight of the submarine, including water in free flooding spaces
W_e	Deviation of the weight of the submarine from W_o
ΔW_e	Artificial noise for filter design
W_f	Weight of flooded or pumped water
W_{int}	Initial value of W before flooding or pumping
W_o	Weight of the submarine when "in trim"
x	Longitudinal body axis or longitudinal coordinate of a point relative to the body axes' origin
$\underline{x}$	Nine dimensional state vector or a state vector in general
$\tilde{\underline{x}}$	Filter estimate error vector
x_B	The x coordinate of the CB
$\underline{x}_c$	Reduced, six dimensional controller state vector
x_f	The x coordinate of the center of mass of flooded or pumped water
$\underline{x}_f$	Reduced, six dimensional, filter state vector
$\hat{\underline{x}}_f$	Filter estimate of $\underline{x}_f$
x_G	The x coordinate of the CG
x_{Ge}	The deviation of x_G from x_{Go}

Δx_{Ge}	Artificial noise for filter design
x_{Gint}	Value of x_G prior to flooding or pumping
x_{Go}	The value of x_G when the submarine is "in trim"
x_o	Inertial longitudinal axis, fixed in a horizontal plane and directed in the foward direction of the submarine's velocity
X	Hydrodynamic force component along x axis (longitudinal, or axial, force)
X_{qq}	Second order coefficient used in representing X as a function of q. First order coefficient is zero
X_t	Resultant force component along the x axis
$X_{\dot{u}}$	Coefficient used in representing X as a function of $\dot{u}$
X_{uu}	Second order coefficient used in representing X as a function of u in the non-propelled case. First order coefficient is zero
X_{wq}	Coefficient used in representing X as a function of the product wq
X_{ww}	Second order coefficient used in representing X as a function of w. First order coefficient is zero
$X_{ww\eta}$	First order coefficient used in representing X_{ww} as a function of $(\eta - 1)$
$X_{\delta b \delta b}$	Second order coefficient used in representing X as a function of δ_b. First order coefficient is zero
$X_{\delta s \delta s}$	Second order coefficient used in representing X as a function of δ_s. First order coefficient is zero

$X_{\delta s \delta s \eta}$	First order coefficient used in representing $X_{\delta s \delta s}$ as a function of $(\eta - 1)$
$X_{\Delta u}$	First order coefficient used in the representation of X as a function of Δu in the linearized EOM
$X_{\Delta w}$	First order coefficient used in the representation of X as a function of Δw in the linearized EOM
$X_{\Delta \Theta e}$	First order coefficient used in the representation of X_t as a function of Θ_e in the linearized EOM
$X_{\Delta \dot{\Theta} e}$	First order coefficient used in the representation of X as a function of $\dot{\Theta}_e$ in the linearized EOM
y	Lateral body axis or the coordinate of a point relative to the body axes
$\underline{y}$	Output vector
$\underline{y}_f$	Noiseless Filter measurement vector
$\hat{\underline{y}}_f$	Filter estimate of $\underline{y}_f$
z	Normal body axis of the submarine or normal component of a point relative to the body axes
z_B	The z coordinate of the CB
$\underline{z}_c$	Measurement signal of the controller state vector $(\underline{x}_c)$
$\underline{z}_f$	Measurement signal of the filter state vector $(\underline{x}_f)$
z_G	The z coordinate of the CG
z_o	A coordinate of the displacement of the CG relative to the origin of the inertial axes

z_{oc}	Command depth, measured relative to the inertial axes
z_{oe}	Deviation of z_o from z_{oc}
Z	Hydrodynamic force component along z axis (normal force)
Z_*	Normal force when body angle (α) and control surface angles are zero
Z_q	First order coefficient used in representing Z as a function of q
$Z_{q\eta}$	First order coefficient used in representing Z_q as a function of (η - 1)
$Z_{\dot{q}}$	Coefficient used in representing Z as a function of $\dot{q}$
$Z_{\lvert q\rvert\delta s}$	Coefficient used in representing $Z_{\delta s}$ as a function of q
Z_t	Resultant force component along the z axis
Z_w	First order coefficient used in representing Z as a function of w
$Z_{w\eta}$	First order coefficient used in representing Z_w as a function of (η - 1)
$Z_{\dot{w}}$	Coefficient used in representing Z as a function of $\dot{w}$
$Z_{\lvert w\rvert}$	First order coefficient used in representing Z as a function of w; equal to zero for symmetrical function
$Z_{w\lvert q\rvert}$	Coefficient used in representing Z_w as a function of q

$Z_{w|w|}$ Second order coefficient used in representing Z as a function of w

$Z_{w|w|\eta}$ First order coefficient used in representing $Z_{w|w|}$ as a function of $(\eta - 1)$

Z_{ww} Second order coefficient used in representing Z as a function of w; equal to zero for symmetrical function

$Z_{\delta b}$ First order coefficient used in representing Z as a function of δ_b.

$Z_{\delta s}$ First order coefficient used in representing Z as a function of δ_s

$Z_{\delta s\eta}$ First order coefficient used in representing $Z_{\delta s}$ as a function of $(\eta - 1)$

$Z_{\Delta u}$ First order coefficient used in the representation of Z as a function of Δu in the linearized EOM

$Z_{\Delta w}$ First order coefficient used in the representation of Z as a function of Δw in the linearized EOM

$Z_{\Delta \dot{w}}$ First order coefficient used in the representation of Z as a function of $\dot{w}$ in the linearized EOM

$Z_{\Delta \ddot{z}oe}$ First order coefficient used in the representation of Z as a function of $\ddot{z}_{oe}$ in the linearized EOM

$Z_{\Delta \dot{\Theta}e}$ First order coefficient used in the representation of Z as a function of $\dot{\Theta}_e$ in the linearized EOM

$Z_{\Delta \ddot{\Theta}e}$ First order coefficient used in the representation of Z as a function of $\ddot{\Theta}_e$ in the linearized EOM

$Z_{\Delta \delta s}$ First order coefficient used in the representation of Z as a function of δ_s in the linearized EOM

α	Angle of attack
α_o	Equilibrium angle of attack
$\underline{\delta}$	Planes deflection vector
δ_b	Deflection of bow or fairwater planes
δ_s	Deflection of stern planes
$\underline{\lambda}$	Eigenvalue matrix
η	The ratio u_c/U
η_o	Equilibrium value of η
ω	Radial frequency
$\underline{\Omega}$	Angular velocity of the body axes
θ	Angle of pitch
θ_c	Command angle of pitch (ordered bubble)
θ_e	Deviation of θ from θ_c
$\cdot$	Symbol placed over a variable denoting the derivative of that variable with respect to time
$\wedge$	Symbol placed over a variable to denote an estimate or a unit vector

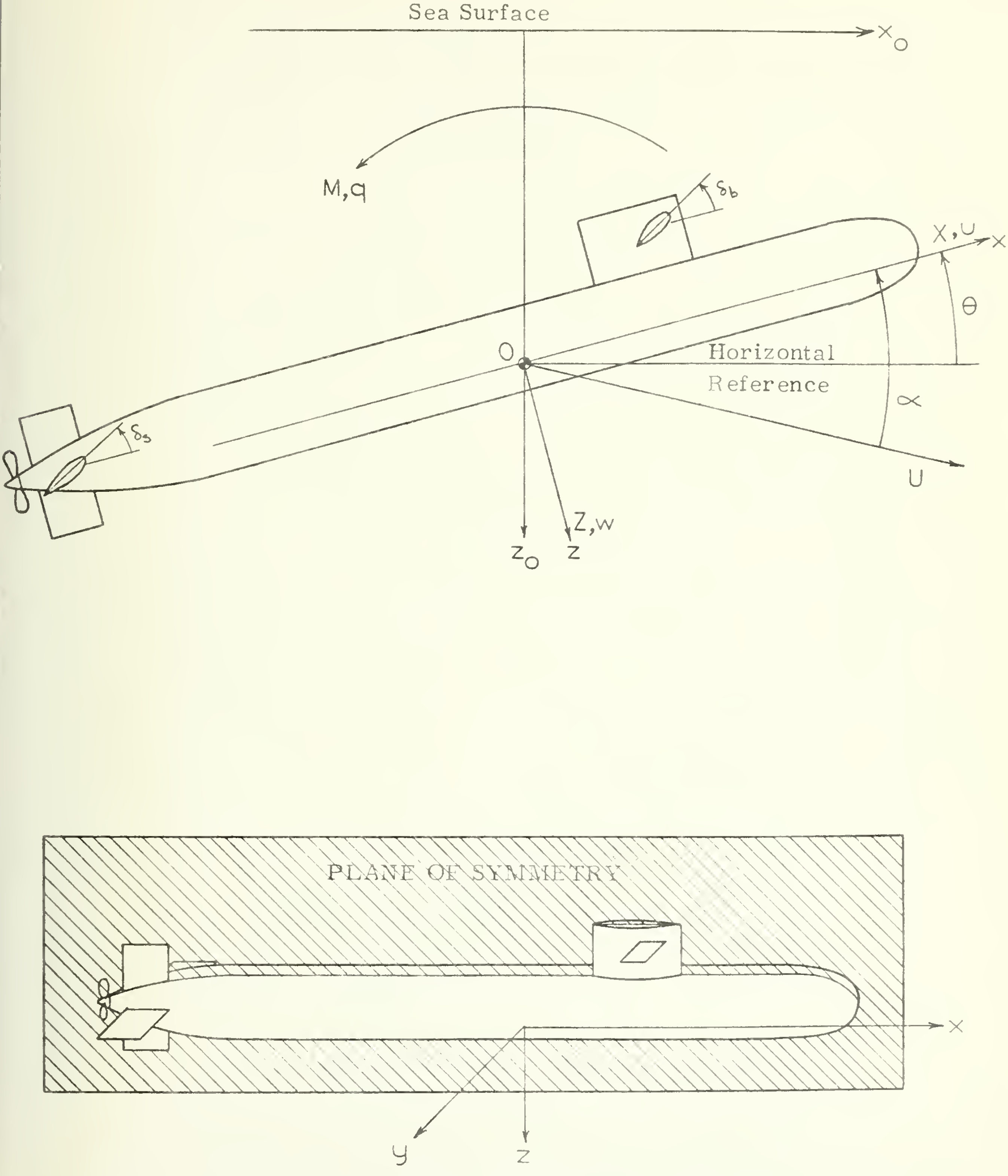

Figure 1. Illustration of Plane of Symmetry and Positive Directions of Axes, Angles, Velocities, Forces and Moment.

Chapter I

Introduction

1. <u>General</u>

Over the past few decades, many technological advances have been made in the design and construction of submarines. They are traveling faster, deeper, and for longer periods of time than ever before. Nontheless, there are many things that remain the same. Naturally, the physical laws governing the behavior of submarines in hydrospace have not been altered by man's more adept manipulations. To be specific submarines are still subjected to hydrostatic and hydrodynamic forces, the magnitude and direction of which are dictated by the designer's appreciation for the natural laws. Another unaltered fact is that these undersea craft are manned by sumbariners whose fates lie largely in their respect for the sea that surrounds them Tragedies of the past bare stark evidence to how intolerant of error the sea can be. Yet it is said, "To err is human." Designers, in their appreciation of this fact, strive for safety through factors of safety and various devices designed to protect the submarine and her crew. Submariner's seek refuge in exhaustive training, practice, and vigilance Indeed, the results have been impressive. It has been the experience of the Author and most submariners that serious mishaps are seldom precipitated by any one mistake or mechanical failure. Post accident investigations most often reveal a chain of events which led to a disturbing, if not disasterous, culmination.

Imagine, for example, a nuclear submarine making a high speed submerged transit. It has been going fast for several hours, and unknown to the crew, it has become grossly heavy, or out of trim as submariner's are apt to say. The pitch angle of the boat and the control surface deflections have been monitored closely, but their slight deviations from neutral angles and the ensuing large hydrodynamic forces remain undetected. Suddenly, main propulsion is lost, and emergency power is channeled to auxiliary systems while the submarine

begins to slow and the engineering department works hurridly to restore main power. At the diving stand, it soon becomes apparent that much greater deflections of the control surfaces and pitch angle are required to maintain depth. The diving officer, realizing that the boat is heavy, gives orders to commence pumping variable ballast to sea. The officer of the deck quickly grasps the gravity of the situation and orders an emergency surfacing. The control surfaces go to maximum deflections in an effort to drive the boat toward the surface while air valves are opened to blow main ballast tanks. Unexpectedly the air lines rupture and emergency power is lost. In the dim light of the battle lanterns panic begins to take hold as the men sense the rising pressure of the atmosphere against their eardrums. Having lost its initial momentum, the submarine reaches an apex and then begins to descend, slowly at first but then more and more rapidly. The officer of the deck has had difficulty making himself understood, but finally the main ballast blow is secured and the air and trim systems are lined up in a desperate effort to blow variable ballast to sea. The process seems to take forever and the diving officer gives the count-down as he announces the increasing depth in ten foot increments. The submarine is well beyond test depth and into the "Jesus Factor" before the lineup is completed and blowing commences. It is unfortunately too little and too late.

It must be admitted that such an occurance is highly unlikely, but statistics over the past eleven years indicate that disasterous chains of events are not impossible. This scenario would have ended differently had the submarine not been out of trim initially. The purpose of this thesis is to derive and test an algorithm for a trim filter, that is, an electronic filter capable of trim analysis.

2. Trim Analysis

In the way of background, trim analysis is one of those facets of submarining that has been affected little by recent technological advances. It is the skill of estimating the required distribution of variable ballast to get the submarine in an "in trim" condition. Such a condition has been achieved when the submarine is capable of maintaining ordered depth with an ordered angle of pitch and with minimal deflections of the control surfaces.

The ordered angle of pitch is most commonly referred to as "ordered bubble" and sometimes ordered trim angle The term bubble derives from the inclinometers used on older submarines. The instruments were inverted "U" shaped glass tubes nearly completely filled with a colored fluid. A small air bubble always found its way to the highest point in the tube. An incremented scale behind the tube indicated the angle at which the instrument was inclined. Since they were aligned in a fore and aft direction, the measured angle (or bubble if you will) coincided with the boat's angle of pitch.

Sometimes, evolutions within the submarine dictate the ordered bubble. For example, while servicing torpedoes, it is frequently necessary to unlash torpedoes in their skids or raise the stop bolts which restrain them in torpedo tubes so that they may be loaded or unloaded. During the time that the weapons are free to travel longitudinally, it is wise to trim for a zero bubble so that the torpedomen need not wrestle unnecessarily with the forces of gravity. However, the prime candidate for ordered bubble is the so called "neutral" bubble. As the submarine changes speed, hydrodynamic forces and moments change in proportion to the velocity squared. When the submarine is at the neutral bubble, these changes are in equilibrium, and it is neither necessary to change the wieght of the submarine nor to shift the center of gravity.

Necessary changes of variable ballast are accomplished with the trim system. Basically, it consists of tanks located throughout the

boat. The forward trim tank is in the vicinity of the bow; after trim tank, near the stern; and the auxiliary tanks near midships, number one to starboard and number two to port. All tanks are connected to a trim manifold for which there are also water lines to and from a trim pump and the sea. Through the correct combinations of opened and closed valves at the manifold, it is possible to control the transfer of variable ballast.

As was implied in the scenario, it is the duty of the diving officer to keep the submarine on ordered depth and, towards that end, to keep the submarine in trim. He observes the actions of the planesmen in their effort to maintain ordered depth and ordered bubble. Due to external disturbances and natural tendencies of the planesmen and the submarine, the planes are seldom stationary, nor is the bubble. The diving officer therefore mentally estimates the averages of those angles. Based upon those average angles, he then makes an estimate of the state of his trim, which is more qualitative than quantitative. If he deems it necessary, he issues commands to the trim manifold operator to correct errors in the submarine's weight (W) or LCG(x_G). Table I illustrates possible states of trim. "Stick" diagrams, such as that

	State	
1.	In Trim	$W = W_o$, $x_G = x_{Go}$
2.	Heavy Overall (HOA)	$W > W_o$
3.	Light Overall (LOA)	$W < W_o$
4.	Heavy Forward (HF)	HOA, $x_G > x_{Go}$
5.	Heavy Aft (HA)	HOA, $x_G < x_{Go}$
6.	Light Forward (LF)	LOA, $x_G < x_{Go}$
7.	Light Aft (LA)	LOA, $x_G > x_{Go}$
8.	Heavy Forward, Light Aft	$W = W_o$, $x_G > x_{Go}$
9.	Light Forward, Heavy Aft	$W = W_o$, $x_G < x_{Go}$
10.	Alright Fore and Aft	$x_G = x_{Go}$, HOA or LOA

Table I. States of Trim

shown in Figure 2, are useful in representing the estimate of averaged angles. For this particular illustration, it is assumed that

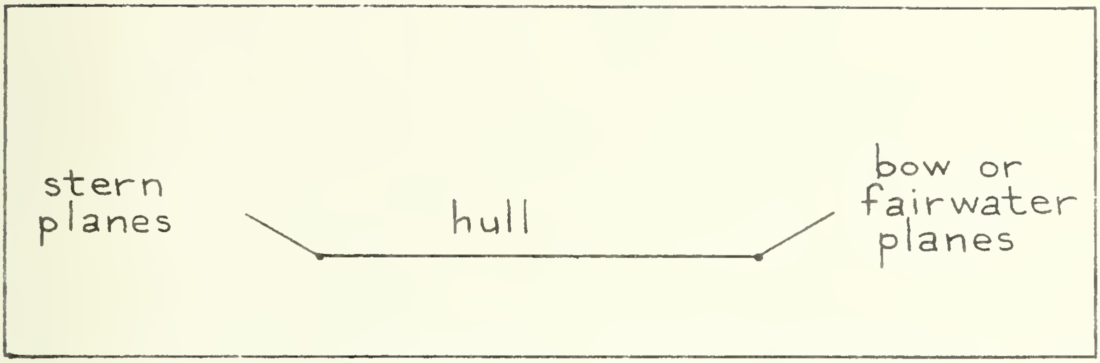

Figure 2. "Stick" Diagram

the pitch angle is near neutral so that it contributes nothing to the state of equlibrium. The fairwater planes impart an upward force and a positive pitching moment, while the stern planes impart a downward force and a positive pitching moment. The net effect is a force near zero in the vertical direction and a positive pitching moment. The submarine is therefore heavy forward, light aft. The correct command would be "Pump from forward trim to after trim." The resulting shift aft of the center of gravity would provide the necessary hydrostatic pitching moment, so the planes should return to zero deflections. During the pumping operation, the diving officer would monitor the planes to ensure his analysis had been correct and to issue the command, "secure pumping" when the submarine was in trim.

3. Factors Affecting a Submarine's Trim

Once the diving officer has achieved the state of bliss known as being in trim, he will not be able to relax for the remainder of his watch. For unfortunately, the state of trim of a submarine is always in a state of change. On the brighter side, the rate of change is usually

slow.

First, there are changes within the submarine. Provisions are consumed, water is distilled and consumed, and sanitary tanks are filled and evacuated. Bilges fill and are pumped. The crew is in a perpetual state of motion. Many an officer student has been victimized by a "trimming party" composed of his classmates. As the student diving officer concentrates under the watchful eye of his instructor, members of the trim party file by inconcpicuously from one end of the boat to the other. Just as orders are given to pump from forward trim to after trim, the trim party commences its migration from the bow to the stern. It is a wise diving officer who monitors planes, angle, and passageway.

Environmental factors affect the submarine's trim also. One relevant variable is the sea water density. As the submarine passes from one density to the next, the hydrostatic buoyant force changes and compensation must be made. Near surface effects constitute another problem area. The boat is subjected to an abundance of external disturbances. There is an apparent suction between the boat and the surface for which compensation must be made. Deviations from mean angles become more severe for the planes and hull. Plane deflection indicators do not reveal the true angles of attack of the planes as nearby surface waves affect water particle velocities about the submarine. Under such circumstances, mental averaging becomes more difficult and trim analysis more clouded As one of Antony's oarsmen lamented to a comrade at the Battle of Actium, "There must be a better way!"

4. A Proposal

The present technique of trim analysis could at best be described as adequate. The advent of nuclear propulsion has highlighted two of its major shortcomings. First, as speed increases, hydrostatic im-

balances become difficult, if not impossible. to perceive. Secondly, despite the increase in demand on their analytical abilities, diving officers find less time to learn and practice the art of trim analysis due the enormous demands imposed by nuclear propulsion. Never has it been so difficult or time consuming to master the art of submarining in all of its aspects.

In the way of improvement, one suggestion has been to design a device which automatically computes averages for the plane deflections and angle of pitch. While this is credible in its simplicity, it falls short of the mark while the submarine is going through transients in depth. It was indeed an exceptional diving officer who could estimate variations in the trim as his boat was changing depth.

The solution offered by this thesis is the trim filter. It is a device which estimates the response of the submarine to bounded disturbances and measured plane deflections. If the actual response differs from the estimate, the filter revises its estimate of the state of trim. The result is a continuous process of trim analysis. As it turns out, a trim filter is not as complex as one might expect

Chapter II

Procedure

1. General

Optimal estimation techniques were investigated to find a method which was capable of the dynamic prediction and estimation mentioned in the previous chapter. The Kalman filter seemed to be an ideal solution. Basically, the procedure used was

1. derivation of a mathematical model,
2. controller design,
3. filter design, and
4. computer simulation.

Most of the details on the above steps appear elsewhere in the thesis. However, a brief description follows.

2. State Vector Model

Kalman filtering requires that the system be described by a linearized model in state vector notation. As described in reference (5), the general form for such a model is

$$d/dt\ \underline{x}(t) = \underline{A}(t)\ \underline{x}(t) + \underline{B}(t)\ \underline{u}(t) \tag{II-1}$$

where

$\underline{x}(t)$ = a state vector of n variables which completely describe the state of the system at any instant in time,

$\underline{u}(t)$ = a control vector of r variables,

$\underline{A}(t)$ = an n X n system coefficient matrix, and

$\underline{B}(t)$ = an n X r control coefficient matrix.

It can be seen that the rows of the state vector model are merely n first order differential equations. The output of the system is described as

$$\underline{y}(t) = \underline{C}(t)\ \underline{x}(t) \tag{II-2}$$

where

$\underline{y}(t)$ = an output vector of m variables, and

$\underline{C}(t)$ = an m X n output coefficient matrix.

The most obvious candidates for variables in the state vector are depth (z_o), descent rate ($\dot{z}_o$), pitch (Θ), pitch rate ($\dot{\Theta}$), and forward velocity (u). Usually, terms related to the mass and center of gravity of such a system are included in $\underline{A}(t)$. However, the nature of the trim problem is that these values are unknown. Rather than use classical parameter estimation techniques to compute the values of the coefficients, it was decided to include the submarine weight (W) and longitudinal center of gravity (x_G) in the state vector. This serves two purposes. First and foremost, Kalman filtering facilitates the estimation of components of the state vector for which no direct measurements are available. Hence, a filter can be used to estimate the unmeasurable variables W and x_G. Secondly, the solutions and simulations are for constant propeller shaft rpm. Since ordered velocity is constant and variations in the trim are accounted for in the state vector, it can be assumed that the coefficient matrices are constant. Hence, equation (II-1) becomes

$$d/dt\ \underline{x}(t) = \underline{A}\ \underline{x}(t) + \underline{B}\ \underline{u}(t) \tag{II-3}$$

and equation (II-2) becomes

$$\underline{y}(t) = \underline{C}\ \underline{x}(t) \tag{II-4}$$

Finally, control surface deflections are included in the state vector. This is to facilitate the controller design since actually the rates of plane deflections are the control variables in depth keeping.

Appendix A describes the details of the derivation and linearization of the equations of motion. This is done about an equilibrium state (Appendix B), so that the components of the state vector are the deviations from equilibrium. The complete state vector is therefore

$$\underline{x} = \begin{bmatrix} Z_{oe} \\ \dot{Z}_{oe} \\ \Theta_e \\ \dot{\Theta}_e \\ \delta_b \\ \delta_s \\ \Delta u \\ x_{Ge} \\ W_e \end{bmatrix} \qquad \text{(II-5)}$$

Notice that equation (II-3) implies that the coefficients of $\underline{\dot{x}}(t)$ on the left hand side is the identity matrix, $\underline{I}$. However, the form of the equations after the linearization is

$$d/dt\ (\underline{V}\,\underline{x}) = \underline{W}\,\underline{x} + \underline{Z}\,\underline{u} + \underline{ZZ}\ \underline{F} \qquad \text{(II-6)}$$

where, using the nomenclature of Appendix A,

$$\underline{F} = \begin{bmatrix} F_{Zs} + F_{Z\ ext} \\ F_{Ms} + F_{M\ ext} \\ F_{Xs} + F_{X\ ext} \end{bmatrix} \qquad \text{(II-7)}$$

$$\underline{u} = \begin{bmatrix} \dot{\delta}_b \\ \dot{\delta}_s \end{bmatrix} \qquad \text{(II-8)}$$

and the coefficient matrices are

$$
\underline{V} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Z_{\Delta\ddot{z}\theta} & 0 & Z_{\Delta\ddot{\theta}\theta} & 0 & 0 & Z_{\Delta\dot{w}}\tan\theta_c & 0 & \frac{w_o}{g} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{M_{\Delta\dot{w}}}{\cos\theta_c} & 0 & I_{yo}-M_{\dot{q}} & 0 & 0 & M_{\Delta\dot{u}} & 0 & M_{\Delta\dot{m}g} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & m_o z_G & 0 & 0 & m_o - X_{\dot{u}} & 0 & \frac{u_o}{g} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\underline{W} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \dfrac{Z_{\Delta w}}{\cos\theta_c} & Z_{\Delta w} u_o & Z_{\Delta\dot{\theta} e} & \cos^2\theta_c Z_{\delta b} & Z_{\Delta\delta s} & Z_{\Delta u} & 0 & \cos\theta_c \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \dfrac{M_{\Delta w}}{\cos\theta_c} & M_{\Delta\theta e} & M_{\Delta\dot{\theta} e} & \cos^2\theta_c M_{\delta b} & M_{\Delta\delta s} & M_{\Delta u} & -W_o\cos\theta_c & M_{\Delta mg} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \dfrac{X_{\Delta w}}{\cos\theta_c} & X_{\Delta\theta e} & X_{\Delta\dot{\theta} e} & 0 & 0 & X_{\Delta u} & 0 & -\sin\theta_c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$\underline{Z} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{II-11}$$

$$\underline{ZZ} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{II-12}$$

Although this format appears cumbersome, it is a must for the steps which follow. The second, fourth and seventh rows of the matrix equations are simply the linearized heave, pitch, and surge equations of Appendix A. In a more straight forward notation, the remaining equations are

$$d/dt \; z_{oe} = \dot{z}_{oe} \tag{II-13}$$

$$d/dt \; \Theta_e = \dot{\Theta}e \tag{II-14}$$

$$d/dt \; \delta_b = \dot{\delta}_b \tag{II-15}$$

$$d/dt \; \delta_s = \dot{\delta}_s \tag{II-16}$$

$$d/dt \; x_{Ge} = 0 \tag{II-17}$$

$$d/dt \; W_e = 0 \tag{II-18}$$

Multiplying both sides of equation (II-6) by the inverse of $\underline{V}$ makes the equation resemble the required form.

$$d/dt\ \underline{x} = \underline{A}\ \underline{x} + \underline{B}\ \underline{u} + \underline{B}'\ \underline{F} \tag{II-19}$$

where

$$\underline{A} = \underline{V}^{-1}\ \underline{W} \tag{II-20}$$

$$\underline{B} = \underline{V}^{-1}\ \underline{Z} \tag{II-21}$$

and

$$\underline{B}' = \underline{V}^{-1}\ \underline{ZZ} \tag{II-22}$$

The last term of equation (II-19) is merely a means of introducing steady state forces and external disturbances during the actual simulations.

3. Depth Controller Design

In order to keep the simulated submarine on ordered depth in the presence of disturbances and to effect depth changes, it was necessary to design a depth controller. Modern optimal control techniques were used rather than classical techniques for two reasons. First, having derived a state vector model of the submarine, the equations were already in the proper format for finding optimal feedback gains. Second, as is demonstrated in the appendices, the optimal control problem and the Kalman filter problem are solved by identical techniques. Hence the Author was able to capitalize on his experience with controller design when designing the filter. Also, the parallels between the two problems were instrumental in grasping the significance of the technique in its application to the filter problem, with which the Author was heretofore unfamiliar.

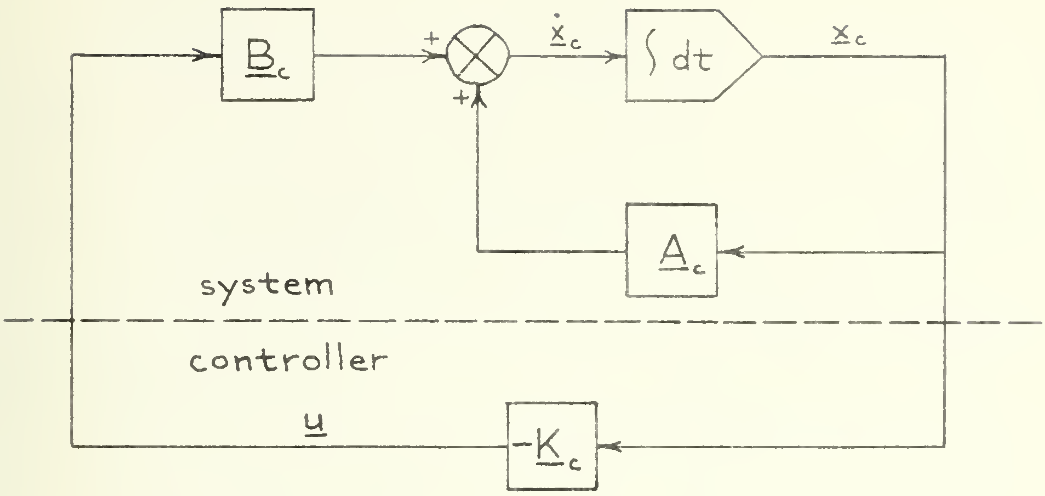

Figure 3. Optimal Controller Structure

Figure 3 illustrates the assumed structure for the controller design. The problem was simplified by not feeding back Δu, x_{Ge}, and W_e. The state vector for the controller design was therefore

$$\underline{x}_c = \begin{bmatrix} Z_{oe} \\ \dot{Z}_{oe} \\ \Theta_e \\ \dot{\Theta}_e \\ \delta_b \\ \delta_s \end{bmatrix} \tag{II-23}$$

and the model for determining controller gains was

$$d/dt\, \underline{x}_c = \underline{A}_c\, \underline{x}_c + \underline{B}_c\, \underline{u} \tag{II-24}$$

where, in terms of the elements of the original matrix, equation (II-20),

$$\underline{A}_c = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \qquad \text{(II-25)}$$

and, as it turns out,

$$\underline{B}_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(II-26)}$$

The details of the manner in which the feedback gains, $\underline{K}_c$, were computed are discussed in Appendices C and E. Stated briefly, a quadratic performance index was selected and the gains emerged from the solution of the matrix Riccati equation. Potter's method was found to be the most effective method of solution, regardless of the weighting matrices in the performance index.

It is re-emphasized that the controller design is not the object of this thesis. It was merely a necessary step in the computer simulation of the submarine dynamics.

4. Filter Design

Figure 4 illustrates the basic relationship between the Kalman filter and the reduced physical system. Detailed descriptions of filters appear in references (5), (7), and (8). The general aspects are discussed below.

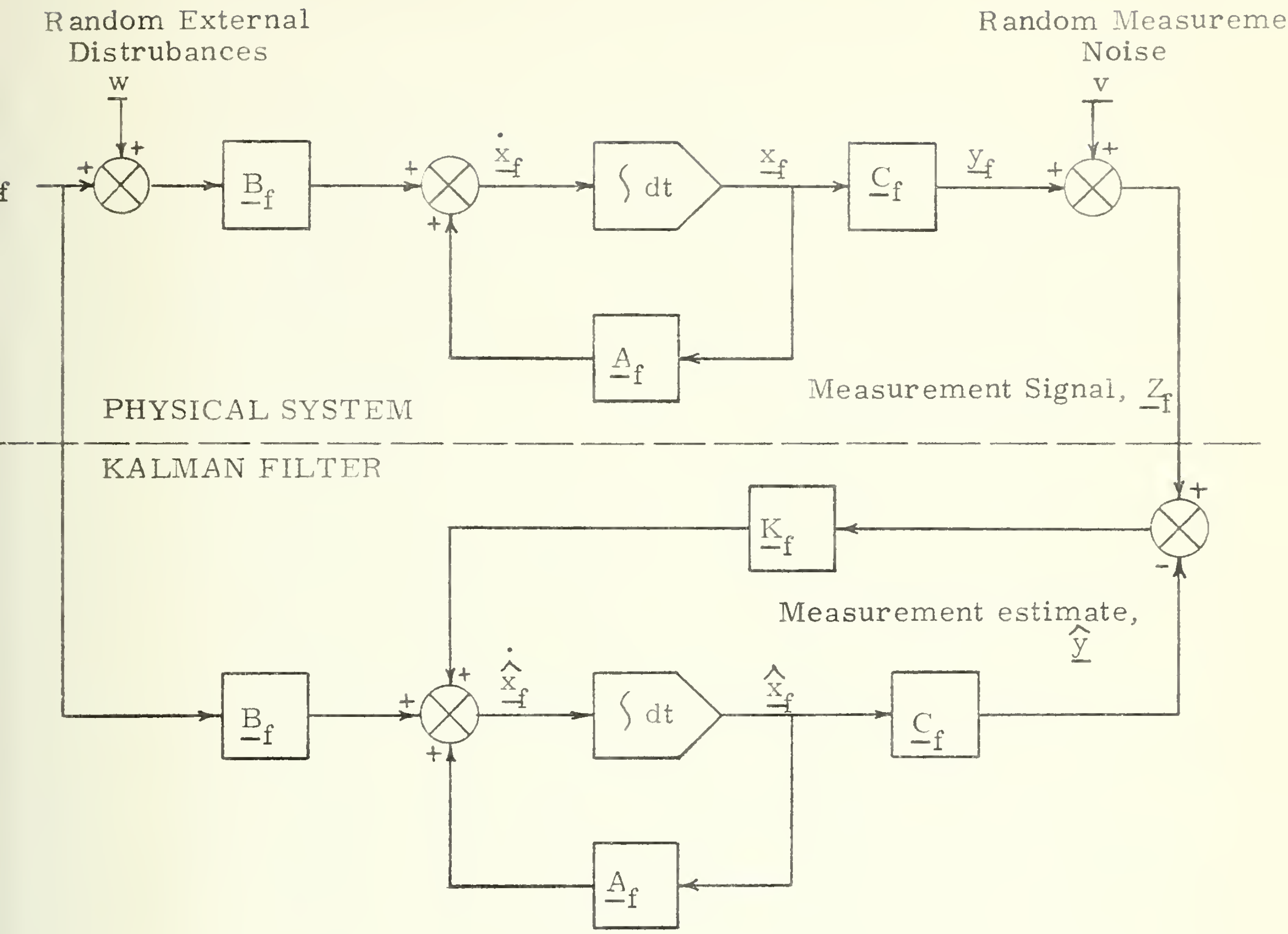

Figure 4. Kalman Filter Structure

The purpose of the Kalman filter is to make a maximum likelihood estimate of the state of a system where variables are inaccessable and/or external and measurement noises are present. The theory is based upon the assumption that the disturbances are in the form of white noise with known statistical properties. This is rather a strict assumption. Its impact on the actual filter depends upon how wide the system's bandwidth is compared with the disturbances'. The system equation is

$$d/dt \ \dot{\underline{x}}_f = \underline{A}_f \underline{x}_f + \underline{B}_f (\underline{u}_f + \underline{w}) \qquad \text{(II-27)}$$

where $\underline{u}_f$ is a deterministic input and $\underline{w}$, a random disturbance. To simplify the filter design, the filter state vector used was

$$\underline{x}_f = \begin{bmatrix} z_{oe} \\ \dot{z}_{oe} \\ \Theta_e \\ \dot{\Theta}_e \\ x_{Ge} \\ W_e \end{bmatrix} \tag{II-28}$$

for which (in terms of equation II-20)

$$\underline{A}_f = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{18} & a_{19} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{28} & a_{29} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{38} & a_{39} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{48} & a_{49} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{88} & a_{89} \\ a_{91} & a_{92} & a_{93} & a_{94} & a_{98} & a_{99} \end{bmatrix} \tag{II-29}$$

Two input vectors were used. The first was

$$\underline{u}_{f1} = \begin{bmatrix} \delta_b \\ \delta_s \\ E(F_{zext}) \\ E(F_{Mext}) \\ \Delta u \end{bmatrix} \tag{II-30}$$

For which(in terms of equations II-20 and 22)

$$\underline{B}_{f1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{25} & a_{26} & b'_{21} & b'_{22} & a_{27} \\ 0 & 0 & 0 & 0 & 0 \\ a_{45} & a_{46} & b'_{41} & b'_{42} & a_{47} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(II-31)}$$

The second input vector used was

$$\underline{u}_{f2} = \begin{bmatrix} \delta_b \\ \delta_s \\ E(F_{Z ext}) \\ E(F_{M\ ext}) \\ \Delta u \\ \Delta x_{Ge} \\ \Delta W_e \end{bmatrix} \qquad \text{(II-32)}$$

For which

$$\underline{B}_{f2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{25} & a_{26} & b'_{21} & b'_{22} & a_{27} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{45} & a_{46} & b'_{41} & b'_{42} & a_{47} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(II-33)}$$

Two sets of variables were assumed to be measurable. The first set is based on the assumption that the submarine is equipped with a modern inertial navigation system capable of measuring z_{oe}, $\dot{z}_{oe}$, Θ_e, and $\dot{\Theta}_e$. The real $\underline{C}_f$ matrix would have gains suitable for converting signal voltages from various instruments to signals corresponding to the actual values of the measured variables. Without jeopardizing the validity

of the results, it was assumed that signals had already been converted so that

$$\underline{C}_{f1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{II-34}$$

The purpose of the second set of variables used was to ascertain the validity of the application of the filter to more conventionally equipped submarines. In such cases there is no instrument for measuring $\dot{z}_{oe}$ directly, so that

$$\underline{C}_{f2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{II-35}$$

These matrices are used to describe the measurement signal, which is

$$\underline{z}_f = \underline{C}_f \underline{x}_f + \underline{v} \tag{II-36}$$

According to theory, if the n X nm composite matrix

$$\left[C^T \mid A^T C^T \mid A^{T2} C^T \mid \ldots \mid A^{T(n-1)} C^T \right] \tag{II-37}$$

is of rank n, then the system is observable, ie., a filter can be desiged. After the system was properly scaled, the above mentioned test was performed successfully for the first input vector ($\underline{u}_{f1}$). The scaling provided for all angles to be measured in degrees, z_{oe} and x_{Ge} in feet, $\dot{\theta}_{oe}$ in degrees/second, $\dot{z}_{oe}$ in feet/second, and W_e in megapounds.

The filter equation is

$$d/dt\ \hat{\underline{x}}_f = \underline{A}_f \hat{\underline{x}}_f + \underline{B}_f\ \underline{u}_f + \underline{K}_f(\underline{z}_f - \hat{\underline{y}}_f) \tag{II-38}$$

where

$$\hat{\underline{y}}_f = \underline{C}_f\ \hat{\underline{x}}_f \tag{II-39}$$

The ideal means of calculating the feedback gains matrix, $\underline{K}_f$, are described in Appendices D and E. These techniques were first applied to a system with input $\underline{u}_{f1}$ (equation II-30). In practice, no solution was possible by the integration technique (time steps as small as 0.01 seconds were attampted), and Potter's technique gave only a partial solution, the first four rows of the gain matrix. Hence, all that would have been possible was estimating the measureable variables. The object of this thesis being to estimate x_{Ge} and W_e, it was essential to find gains for the fifth and sixth rows of $\underline{K}_f$. This was done by using the positive results from Potter's method, supplemented by educated guesses. The technique used is described in Appendix D. It was basically a pole placement technique. The final result was a filter that was two thirds optimal and one third "tweaked."

Subsequently, it was discovered that the problem was amenable to Potter's method if the second input vector, $\underline{u}_{f2}$ (equation II-32), was used. The inclusion of the artifical noises, Δx_{Ge} and ΔW_e, provided a mathematical means for the variation of the related parameters. Hence, gains could be calculated for compensation using the "optimal" technique.

5. Simulation

All simulations were performed on a IBM 360 computer. After the matrices had been set up, the differential equations were solved by a fourth order Runge-Kutta method. Hamming's modified predictor - corrector method was attempted first, but variables whose differentials were zero tended to "drift" rather than remain constant as they should have. A time step of 0.1 seconds was used to make variables changed

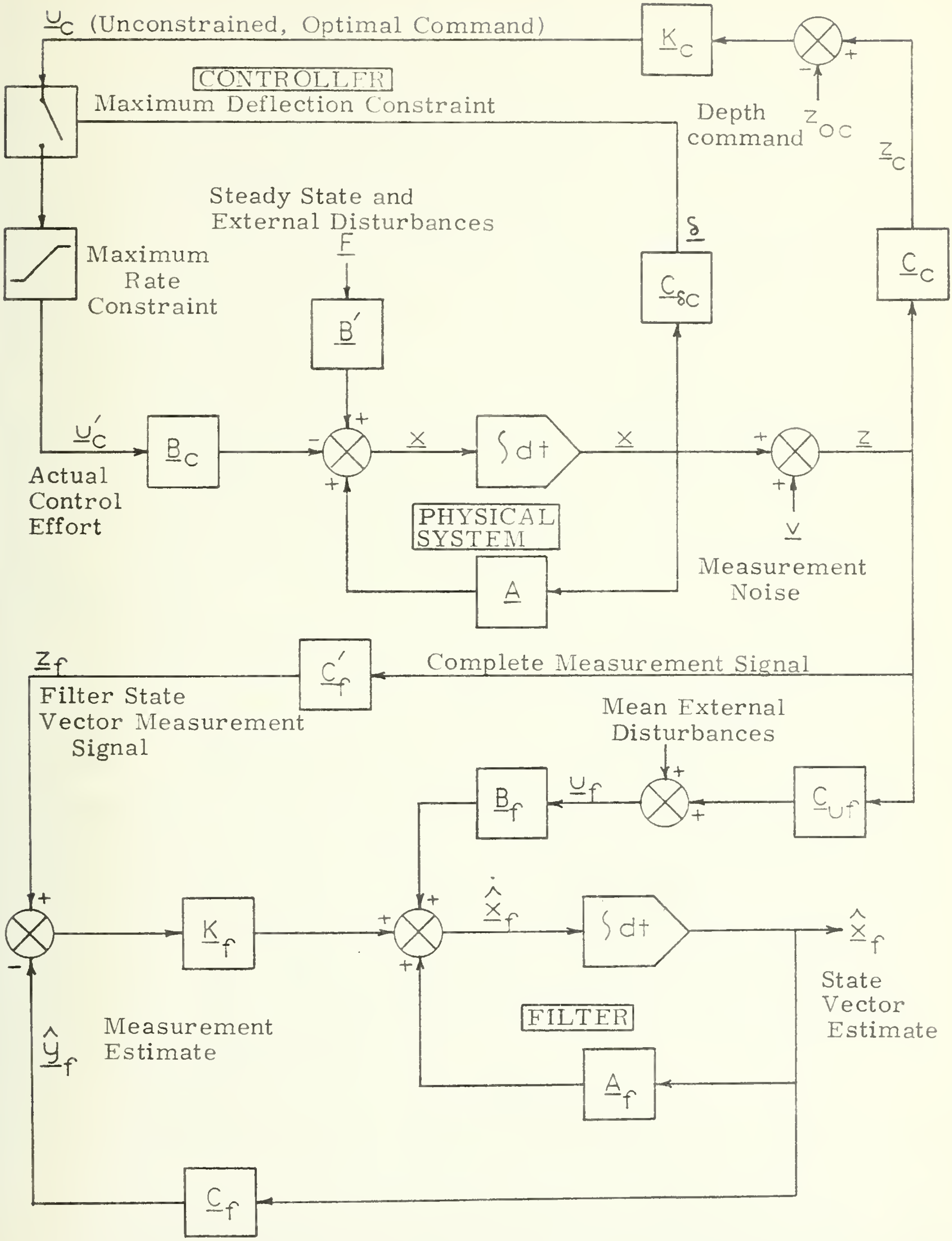

Figure 5. Structure of Physical System Model, Constrained Controller, and Filter for Simulation

between time steps, such as controller or filter feedback and disturbances, behave in a manner closely resembling continuous functions.

The linearized equations of motion were used to simulate the submarine dynamics, and the density of the sea water and the acceleration due to gravity were assumed to remain constant. Figure 5 depicts a block diagram of the simulation program which appears in Appendix F. The gains matrices, $\underline{K}_c$ and $\underline{K}_f$, were calculated as described in the two previous sections. Both rate and position constraints were imposed on the planes.

Preliminary simulations were for the purpose of testing controller gains for given depth changes. Filter gains were tested for step inputs without and with Gaussian noise. Sensitivity to a ramp input (flooding casualty) was tested and a least squares technique for smoothing was tested. All estimates were made while the submarine was periodically changing depth.

The coefficients used appear in Table II. They were derived by manipulating data from a motley collection of books, reports, and articles on fluid drag, potential theory, bodies of revolution, airfoils, submersibles, airships and naval propulsion. Although they represent no real submarine, the results obtained with them seem realistic, based upon the experience of the Author. It is therefore assumed that they are satisfactory for the sake of a numerical example and that the procedures described herein could be successfully applied to an actual set of submarine coefficients. The means of converting from non-dimensional form can be found in references (2) or (3) or in the computer program for simulations in Appendix F.

ℓ	370 ft	$M'_{\lvert w \rvert q}$	0
ρ	1.9924 slugs/ft^3		
m'	.0102	M'_{ww}	-.001
I'_y	.00054	$M'_{w\lvert w \rvert}$	-.004
a_i	.0011	$M'_{w\lvert w \rvert \eta}$	0
b_i	-.0004	M'_{*}	.00006
c_i	.0015	$M'_{\delta b}$	.0008
x'_B	0	$M'_{\delta s}$	-.0025
z'_B	-.003	$M'_{\delta s \eta}$	0
x'_G	0	Z'_q	-.0046
z'_G	0	$Z'_{q\eta}$	0
$X'_{\dot{u}}$	-.00013	$Z'_{\dot{q}}$	-.0002
X'_{uu}	-.0022	Z'_w	-.013
X'_{wq}	-.005	$Z'_{w\eta}$	0
X'_{ww}	-.005	$Z'_{\dot{w}}$	-.0085
$X'_{ww\eta}$	0	$Z'_{\lvert w \rvert}$	.0002
M'_q	-.0028	Z'_{ww}	.001
$M'_{q\eta}$	0	$Z'_{w\lvert w \rvert}$	-.03
$M'_{\dot{q}}$	-.0005	$Z'_{w\lvert w \rvert \eta}$	0
M'_w	.0029	Z'_{*}	-.0003
$M'_{w\eta}$	0	$Z'_{\delta b}$	-.003
$M'_{\dot{w}}$	-.00012	$Z'_{\delta s}$	-.005
$M'_{\lvert w \rvert}$	.0002	$Z'_{\delta s \eta}$	0

Table II. Coefficients & Parameters for Simulations

Chapter III

Results

I. "In Trim" Condition

Figures 6 and 7 depict the solutions of equations (B-14) and (B-16) for W_o and x_{Go} as a function of command speed (u_c) and ordered bubble (Θ_c). It can be seen in figure 6 that a horizontal line drawn through W-B = 0 would lie between $\Theta_c = -1^\circ$ and $\Theta_c = -2^\circ$, hence

$$-2^\circ < \Theta_{neutral} < -1^\circ \qquad \text{(III-1)}$$

Looking at figure 7, it is seen that all curves are parabolic and that those in the vicinity of the bound expressed in (III-1) intersect at about 7.15 knots. It is safe to assume that the curve for the neutral bubble would be a horizontal line passing through the intersection at 7.15 knots. Therefore

$$x_{G\ neutral} = 0.025 \text{ ft} \qquad \text{(III-2)}$$

At zero speed where the only forces are hydrostatic, it can be seen that for each degree of trim, there corresponds a shift in x_G of about .0185 feet. This fact can be used to linearly interpolate between -1° and -2°:

$$\Theta_{neutral} = -1^\circ - \left(\frac{x_{Gneutral} - x_{G1}}{x_{G2} - x_{G1}} \right) \qquad \text{(III-3)}$$

or

$$\Theta_{neutral} = -1^\circ - \left(\frac{0.025 - 0.0195}{0.039 - 0.0195} \right) = -1.28^\circ \qquad \text{(III-4)}$$

For the simulations, it was decided to use the nearest integer to the neutral bubble for ordered bubble, hence

$\Theta_c = -1^\circ$ for simulations.

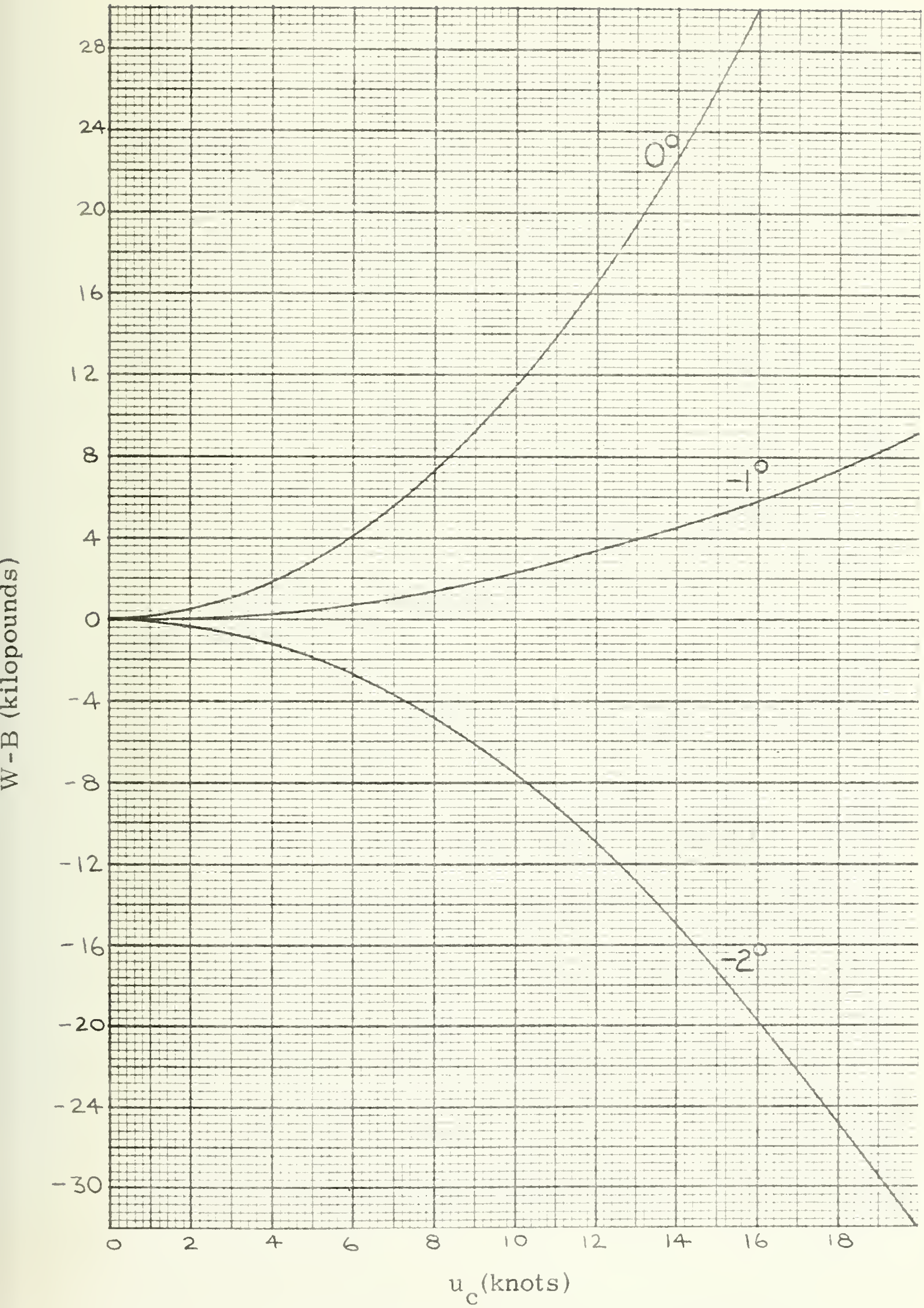

Figure 6. Weight Minus Buoyancy vs. Command Velocity (u_c)

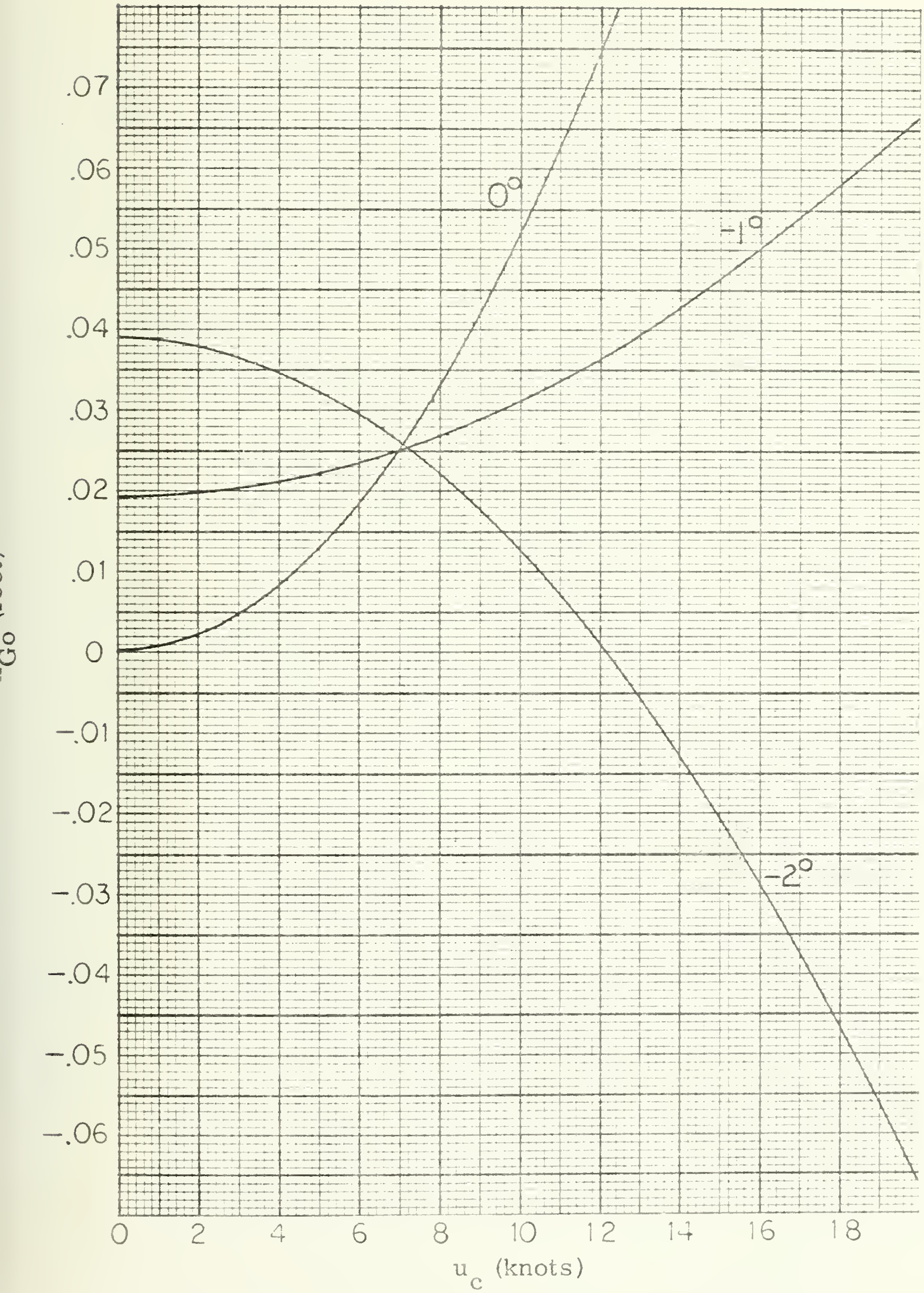

Figure 7. LCG vs. Command Velocity (u_c)

Figure 7 is useful to demonstrate another point. Below 7.15 knots, trimming to pitch the bow down requires shifting the center of gravity forward as one might expect. However, above that speed the opposite is true. This is due to the domination of hydrodynamic moment over the hydrostatic moment. Although the transition is gradual, it can be said that, in general, hydrostatics dominate below 7.15 knots and that hydrodynamics govern above 7.15 knots.

Since trim analysis is most difficult when hydrodynamics dominate, it was decided to test the filter well above the "transition" velocity. Therefore a command speed of 12 knots was used for all the simulations.

2. Controller Tests

Two sets of controller gains were calculated. The weighting matrices for the first were:

$$\underline{Q}_{cl} = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(III-5)}$$

and

$$\underline{P}_{cl} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(III-6)}$$

and the resulting gain matrix was

$$\underline{K}_{cl} = \begin{bmatrix} 8.721 & 43.14 & -3.579 & -34.53 & -.6673 & -.008933 \\ 4.894 & 17.82 & 1.497 & 1.913 & -.008932 & -.4408 \end{bmatrix} \qquad \text{(III-7)}$$

Figure 8 depicts the response of the "controlled" submarine to an order to decrease depth by 50 feet. When last seen, the submarine was passing 4000 feet with a pitch angle of -116° ! Crews have jokingly given diving officers quarters for much less exciting rides.

Going back to the weighting matrices, the logic in $\underline{Q}_{c1}$ was that maintaining depth was far more important than ordered bubble which was the only other variable worth weighing. Most diving officers would have pitched up about 5° for such a depth change, but the "optimal" controller which is actually a linear regulator does not account for the constraints on the planes' deflections. Even though the error in depth far exceeds the bubble error, the controller "thinks" it can change depth in an optimal manner while maintaining a pitch angle close to the ordered bubble by using plane deflections exceeding 180° if the need exists. This highlights two short-comings: the failure to account for constraints and the failure of linear theory to account for the "stalling" of control surfaces at large angles of attack.

By the time Θ_e became large, z_{oe} became much larger and was still the predominant error. With luck, this controller might have accomplished a depth change of 8 feet.

An attempt was made to calculate a set of gains for which there was no weighing of pitch angle error, however, the Riccati equation was unsolvable for such a case which the techniques at the disposal of the Author. Therefore a compromise was made for the second set of gains. Θ_e was weighted more heavily as was $\underline{u}_c$. The matrices were:

$$\underline{Q}_{c2} = \begin{bmatrix} 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 400 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(III-8)}$$

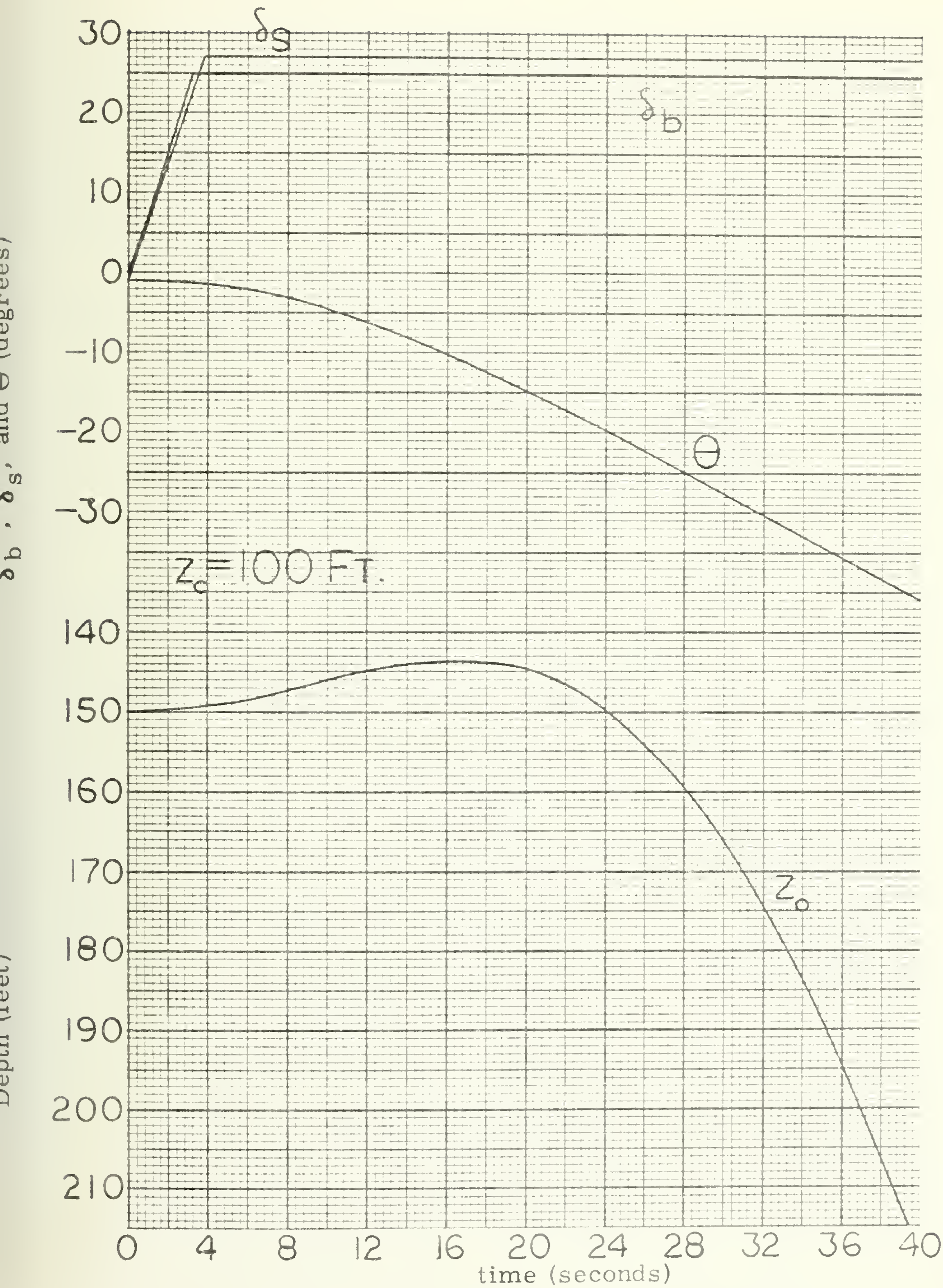

Figure 8. $\underline{K}_{cl}$ Depth Controller Simulation

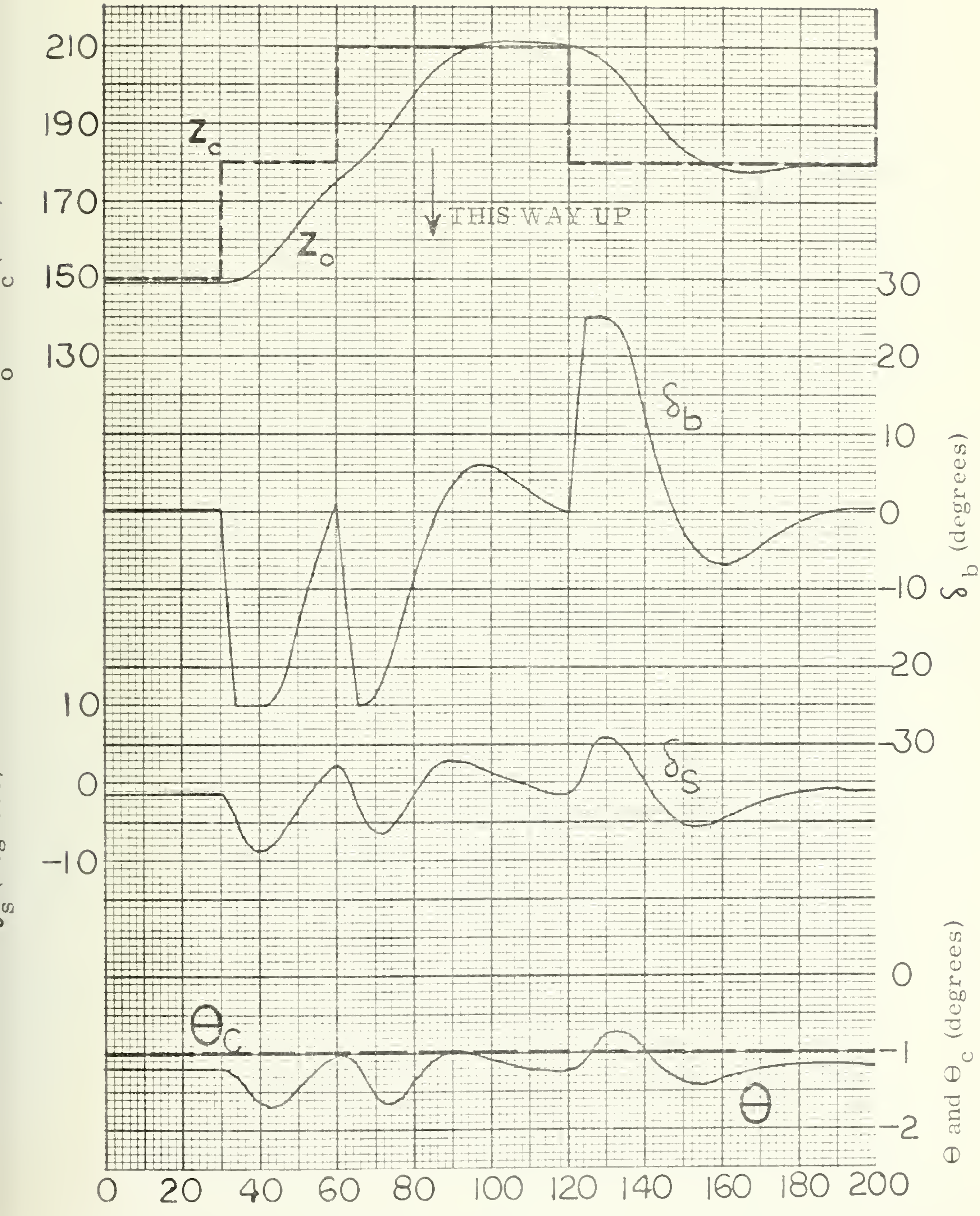

Figure 9. $\underline{K}_{c2}$ Depth Keeping History for Simulations

$$\underline{P}_{c2} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \qquad \text{(III-9)}$$

and

$$\underline{K}_{c2} = \begin{bmatrix} .2999 & 2.996 & -.7980 & -6.049 & -.2036 & .082163 \\ .0068262 & 1.103 & 2.147 & 8.762 & .082163 & -.3585 \end{bmatrix}$$

(III-10)

Comparing $\underline{K}_{c2}$ with $\underline{K}_{c1}$, one can see the decrease in the total control effort and the increase in the relative importance of Θ_e over z_{oe}. Figure (9) shows the typical record of depth change orders and submarine response for the remainder of the simulations.

3. <u>Filter Gains</u>

Four filter feedback gains matrices ($\underline{K}_f$) were calculated. The matrices and relevant data appear in tables III thru VI. Unless indicated otherwise, off-diagonal elements of square matrices are zero.

Also note that δ_b, δ_s, and Δu are treated as external disturbances. The noise assigned to these variables can be used to imply the uncertainty which exists in their physical impact on the submarine dynamics. For example, it is the angle of attack of the control surfaces, rather than their deflections relative to the submarine, which determines the forces and moments they impart on the submarine. The greatest degree of uncertainty for these angles exists near the surface.

Table III $\underline{K}_{f1}$ and Related Data

Speed: 12 knots; Ordered Bubble: -1°

Input Vector: $\underline{u}_{f1}$ (equation II-30)

Measurement Matrix: $\underline{C}_{f1}$ (equation II-34)

Technique: Potter/pole placement

Variable	E(w or v)	Standard Deviation	$E(w^2 \text{ or } v^2)$
δ_b	0	.1772 degrees	$\pi/100$
δ_s	0	.1772 degrees	$\pi/100$
FZext	0	1,772.45 pounds	$\pi \times 10^6$
FMext	0	17,724.5 foot pounds	$\pi \times 10^8$
Δu	0	.1063 feet/second	.0113
z_{oe}	0	.8862 feet	.7854
$\dot{z}_{oe}$	0	.8862 feet/second	.7854
θ_e	0	.1772 degrees	$\pi/100$
$\dot{\theta}_e$	0	.1772 degrees/second	$\pi/100$

$$\underline{Q}_{f1} = \begin{bmatrix} \pi/100 & & & & \\ & \pi/100 & & & \\ & & \pi \times 10^6 & & \\ & & & \pi \times 10^8 & \\ & & & & .0113 \end{bmatrix}$$

$$\underline{P}_{f1} = \begin{bmatrix} .7854 & & & \\ & .7854 & & \\ & & \pi/100 & \\ & & & \pi/100 \end{bmatrix}$$

$$\underline{K}_{f1} = \begin{bmatrix} 0.052295000 & 0.0015072000 & -.0832560 & 0.002256200 \\ 0.001507200 & 0.0001506800 & -.0106650 & -.000061439 \\ -.003330200 & -.0004266000 & 0.0485490 & 0.001320300 \\ 0.000090246 & -.0000024576 & 0.0013203 & 0.000225530 \\ 0.000450000 & 0.0000200000 & -0020000 & -004500000 \\ 0.000030000 & 0.0003200000 & 0.0000000 & 0.000000000 \end{bmatrix}$$

Eigenvalues

Real Part	Imaginary Part
-.1287	0.0
-.03371	.06354
-.03371	-.06354
-.04259	.0353
-.04259	-.0353
-.01234	0.0

Table IV $\underline{K}_{f2}$ and Related Data

Speed: 12 knots; Ordered Bubble: - 1°

Input Vector: $\underline{u}_{f1}$ (equation II-30)

Measurement Matrix: $\underline{C}_{f1}$ (equation II-34)

Technique: Potter/pole placement

Variable	E(w or v)	Standard Deviation	E(w^2 or v^2)
δ_b	0	1 degree	1
δ_s	0	1 degree	1
FZext	0	1 pound	1
FMext	0	1 foot pound	1
Δu	0	1 foot/second	1
z_{oe}	0	.03162 feet	.001
$\dot{z}_{oe}$	0	.03162 feet/second	.001
θ_e	0	.03162 degrees	.001
$\dot{\theta}_e$	0	.03162 degrees/second	.001

$$\underline{Q}_{f2} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

$$\underline{P}_{f2} = \begin{bmatrix} .001 & & & \\ & .001 & & \\ & & .001 & \\ & & & .001 \end{bmatrix}$$

$$\underline{K}_{f2} = \begin{bmatrix} 0.470800 & 0.118200 & -.018969 & 0.022660 \\ 0.118200 & 0.075907 & -.043117 & 0.012530 \\ -.018969 & -.043117 & 0.501400 & 0.136100 \\ 0.022660 & 0.012530 & 0.136100 & 0.099234 \\ 0.003000 & 0.002000 & -.032000 & -.450000 \\ 0.004000 & 0.016000 & 0.000000 & 0.000000 \end{bmatrix}$$

Filter Eigenvalues

Real Part	Imaginary Part
-.3287	.3760
-.3287	-.3760
-.1813	.2268
-.1813	-.2268
-.2430	0.0
-.07681	0.0

Table V $\underline{K}_{f3}$ and Related Data

Speed: 12 knots; Ordered Bubble: -1°

Input Vector: $\underline{u}_{f2}$ (equation II-32)

Measurement Matrix: $\underline{C}_{f1}$ (equation II-34)

Technique: Potter

Variable	E(w or v)	Standard Deviation	$E(w^2 \text{ or } v^2)$
δ_b	0	2 degrees	4
δ_s	0	1 degree	1
FZext	0	10^5 pounds	10^{10}
FMext	0	10^7 foot pounds	10^{14}
Δu	0	2 feet/second	4
Δx_{Ge}	0	0.1 feet	.01
ΔW_e	0	0.05 megapounds	.0025
z_{oe}	0	0.5 feet	.25
$\dot{z}_{oe}$	0	0.2 feet/second	.04
θ_e	0	0.1 degrees	.01
$\dot{\theta}_e$	0	0.1 degrees/second	.01

$$\underline{Q}_{f3} = \begin{bmatrix} 4 & & & & & & \\ & 1 & & & & & \\ & & 10^{10} & & & & \\ & & & 10^{14} & & & \\ & & & & 4 & & \\ & & & & & .01 & \\ & & & & & & .0025 \end{bmatrix}$$

$$\underline{P}_{f3} \begin{bmatrix} .25 & & & \\ & .04 & & \\ & & .01 & \\ & & & .01 \end{bmatrix}$$

$$\underline{K}_{f3} = \begin{bmatrix} 0.38940000 & 0.771500 & -.0046404 & 0.022983 \\ 0.12340000 & 0.801800 & -.0800630 & -.087767 \\ -.00018560 & -.020016 & 0.8350000 & 0.451200 \\ 0.00091933 & -.021942 & 0.4512000 & 0.677100 \\ 0.00481060 & 0.033007 & -.5456000 & -.835100 \\ 0.02275400 & 0.242600 & 0.035506 & 0.084360 \end{bmatrix}$$

Table VI $\underline{K}_{f4}$ and Related Data

Speed: 12 knots; Ordered Bubble: -1°

Input Vector: $\underline{u}_{f2}$ (equation II-32)

Measurement Matrix: $\underline{C}_{f2}$ (equation II-35)

Technique: Potter

Variable	E(w or v)	Standard Deviation	$E(w^2$ or $v^2)$
δ_b	0	2 degrees	4
δ_s	0	1 degree	1
FZext	0	10^5 pounds	10^{10}
FMext	0	10^7 foot pounds	10^{14}
Δu	0	2 feet/second	4
Δx_{Ge}	0	0.1 feet	.01
ΔW_e	0	0.05 megapounds	.0025
z_{oe}	0	0.5 feet	.25
θ_e	0	0.1 degrees	.01
$\dot{\theta}_e$	0	0.1 degrees/second	.01

$$\underline{Q}_{f4} = \begin{bmatrix} 4 & & & & & & \\ & 1 & & & & & \\ & & 10^{10} & & & & \\ & & & 10^{14} & & & \\ & & & & 4 & & \\ & & & & & .01 & \\ & & & & & & .0025 \end{bmatrix}$$

$$\underline{P}_{f4} = \begin{bmatrix} .25 & & \\ & .01 & \\ & & .01 \end{bmatrix}$$

$$\underline{K}^{*}_{f4} = \begin{bmatrix} 0.9680000 & 0.0 & -.048367 & 0.082643 \\ 0.4687000 & 0.0 & -.132600 & -.064191 \\ -.0019347 & 0.0 & 0.836400 & 0.451900 \\ 0.0033057 & 0.0 & 0.451900 & 0.678400 \\ 0.0113010 & 0.0 & -.546800 & -.835400 \\ 0.0998360 & 0.0 & 0.019525 & 0.020985 \end{bmatrix}$$

*The actual dimension of $\underline{K}_{f4}$ are 6 X 3. The above format is required for the computer simulation program (Appendix F). The column of zeroes implies that $\dot{z}_{oe}$ is not measured and that $\hat{\dot{z}}_{oe}$ is not fed back to the filter.

4. Filter Simulations

Filter simulation results are grouped by gains matrices. Responses to step and ramp (flooding) inputs were simulated. The step inputs in W_e were plus or minus 11,000 pounds. Table VII illustrates the comparison of this error to various types of displacements for the test submarine. Other than noise-free

Classification of Displacement	Displacement (tons)	W_e/Displacement
Hydrodynamic (incl. free flooding)	7392	.0746%
Submerged	7170	.0767%
Surfaced	6100	.0902%

Table VII. Comparison of W_e Step to Displacement

simulations to determine the natural responses of the filters, two sets of noises were used to provide a common base for comparing responses. The statistical properties of these Gaussian disturbances appear in tables VIII and IX.

External			Measurement		
Variable	E(w)	Standard Deviation	Variable	E(v)	Standard Deviation
FZext	0	1,000 lb	z_{oe}	0	.5 ft
FMext	0	10,000 ft-lb	$\dot{z}_{oe}$	0	.5 ft/sec
FXext	0	500 lb	Θ_e	0	.1
			$\dot{\Theta}_e$	0	.1 /sec
			δ_b	0	.1
			δ_s	0	.1
			Δu	0	.06 ft/sec

Table VIII. Disturbance Statistics

External Variable	E(w)	Standard Deviation	Measurement Variable	E(v)	Standard Deviation
FZext	0	1 lb	z_{oe}	0	.0316 ft
FMext	0	1 ft-lb	$\dot{z}_{oe}$	0	.0316 ft/sec
FXext	0	1 lb	θ_e	0	.0316
			$\dot{\theta}_e$	0	.0316 /sec
			ζ_b	0	1
			ζ_s	0	1
			Δu	0	.0113 ft/sec

Table IX. Disturbance Statistics

Figure 10 and 11 illustrate the natural responses of filters equipped with $\underline{K}_{f1}$ (Table III) and $\underline{K}_{f2}$ (Table IV) respectively. It can be seen that the response of $\underline{K}_{f1}$ is much slower than $\underline{K}_{f2}$. The disturbances of Table VIII exceed those for which $\underline{K}_{f2}$ was designed for the most part, but are mostly less than those for which $\underline{K}_{f1}$ was designed. Comparing the responses (figures 10, 12, and 13) it can be seen that the slower filter ($\underline{K}_{f1}$) is more stable. Note also, that even though the faster filter ($\underline{K}_{f2}$) is underdesigned, the means of the estimates converge toward the actual trim errors. This suggests some sort of smoothing might be used to salvage the output of the filter. The relatively large overshoots of $\hat{x}_{Ge}$ for the slower filter ($\underline{K}_{f1}$) (figure 10) can be attributed to lack of damping. This is also reflected in the eigenvalues (Table III) for $\underline{K}_{f1}$. The imaginary parts of one pair are nearly twice as large as the real parts.

The disturbances of Table IX were used for subsequent simulations. They are the maximum distrubances for which $\underline{K}_{f2}$ was designed, and this filter's response appears in figure 14. The coupling between $\hat{x}_{Ge}$ and $\hat{W}_e$ can also be seen. "Undershoots" in $\hat{W}_e$ lag overshoots of $\hat{x}_{Ge}$ by about ten seconds.

Figures 15 thru 18 were plotted for a filter with $\underline{K}_{f3}$. An attempt

to plot the response of $\underline{K}_{f4}$ to the same conditions (step input) revealed that the differences between the two filters could not be discerned with the scales used for the graphs. The response of these filters (figure 16) is much faster than the previous two filters. In fact, the new filters are faster than the submarine in a sense. At 30 and 60 seconds, depth change commands were given, causing the planes to deflect. Before the submarine could respond to the planes, the filter "assumed" the plane deflections were due to steps in trim error. Hence, the apparent disturbances of the noiseless simulations at those times. This might also explain the fluctuations for the filter with $\underline{K}_{f2}$ prior to settling out (figure 11).

When subjected to the disturbances of Table IX, the two new filters suffered from relatively severe fluctuations (figure 15) which were smoothed somewhat using a least-squares smoother (figure 16). Data from the previous thirty seconds was used for smoothing estimates. The noise was less than that for which the filters were designed. The smoothed estimate of x_{Ge} lags the unsmoothed estimate by about six seconds while that for W_e lags the unsmoothed estimate by only one second. This might be attributed to the fact that the response of $\hat{W}_e$ is faster than that of $\hat{x}_{Ge}$ (see figure 16).

A filter equipped with $\underline{K}_{f3}$ was subjected to a flooding casualty (figures 17 and 18) of 500 pounds/second, from 5 to 60 seconds, at 125 feet aft of the body axes' reference. The simulation was begun at -30 seconds with zero estimate errors to ensure the filter had "settled out" prior to the casualty. The smoothed estimate of x_{Ge} lags the beginning of the casualty by about 12 seconds and gets within 5 seconds, while that of W_e lags by 5 seconds at first and gets within 3 seconds, excluding the pertubation at 35 seconds due to the plane deflections.

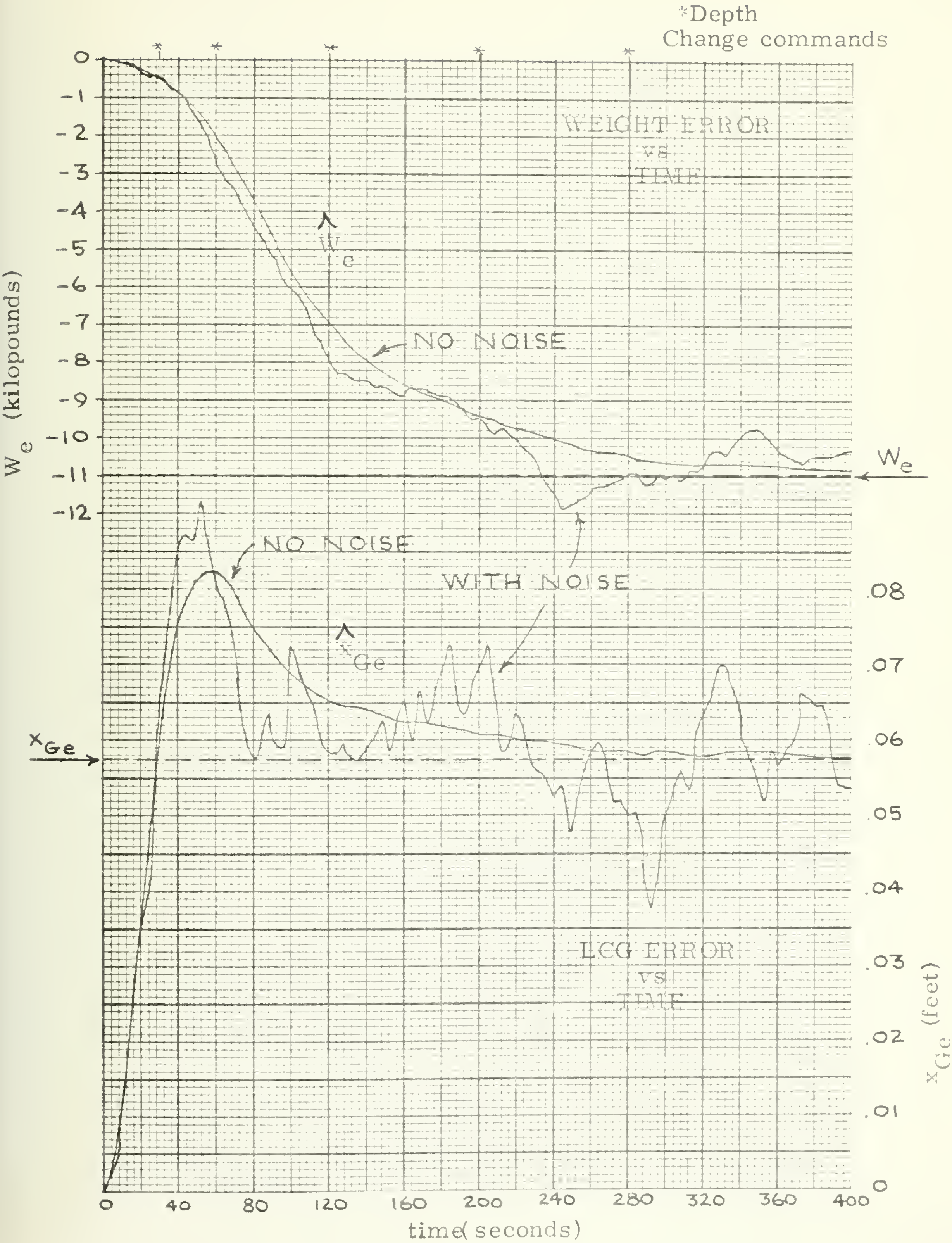

Figure 10. K_{f1}, Step Input, No Noise and Noise (Table VIII)

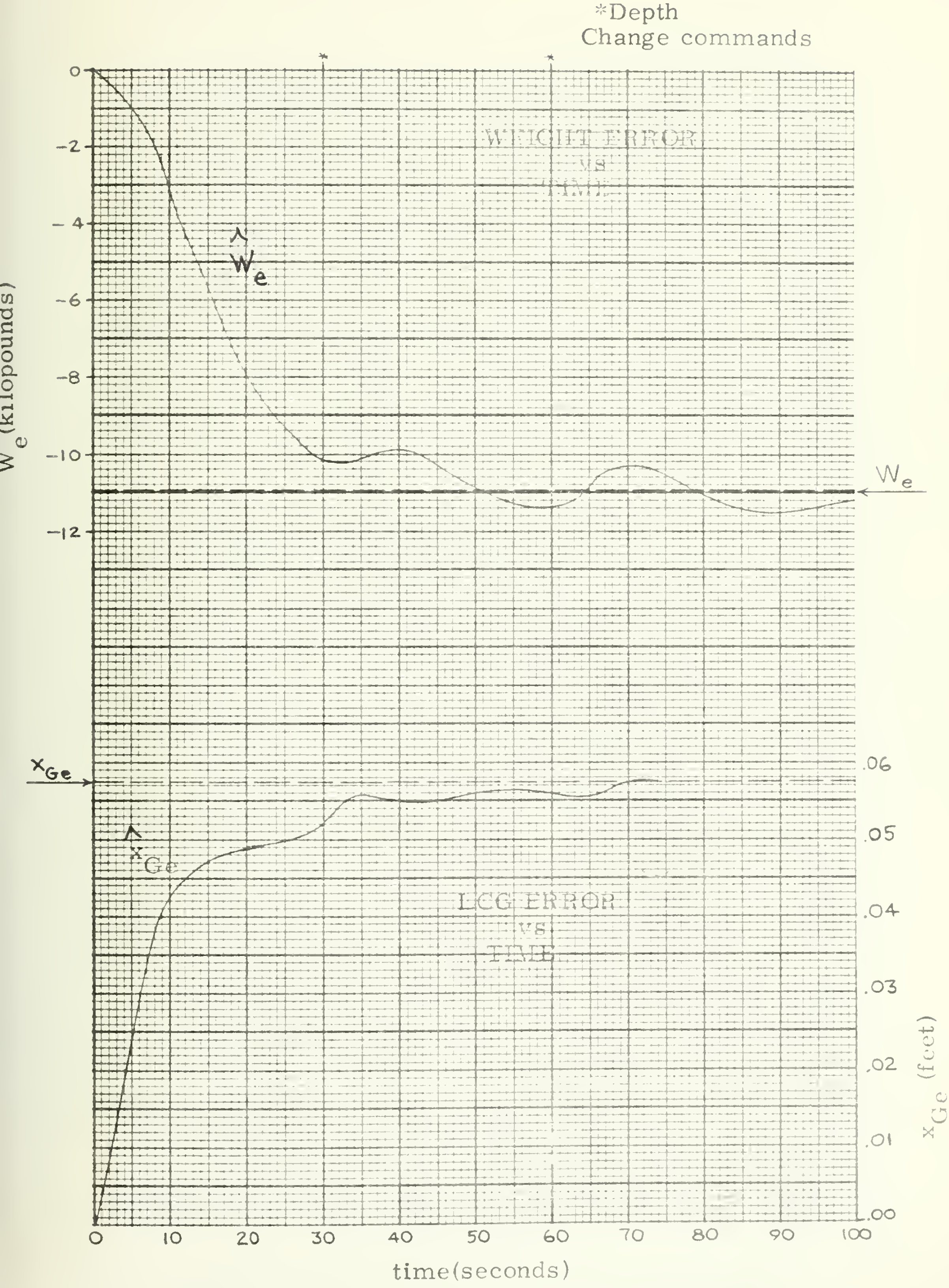

Figure 11. $\underline{K}_{f2}$, Step Input, No Noise

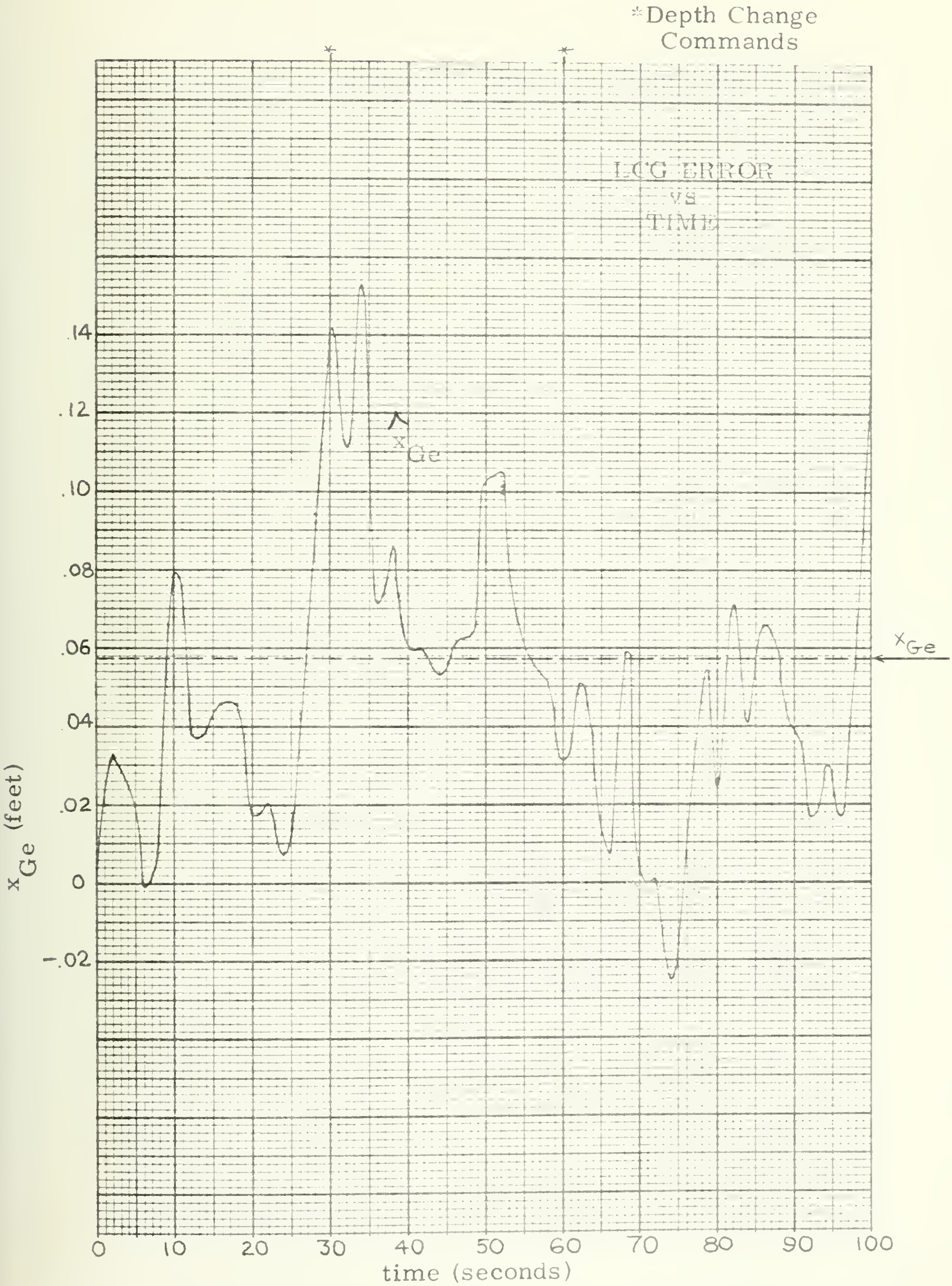

Figure 12. K_{f2}, Step Input, Noise (Table VIII)

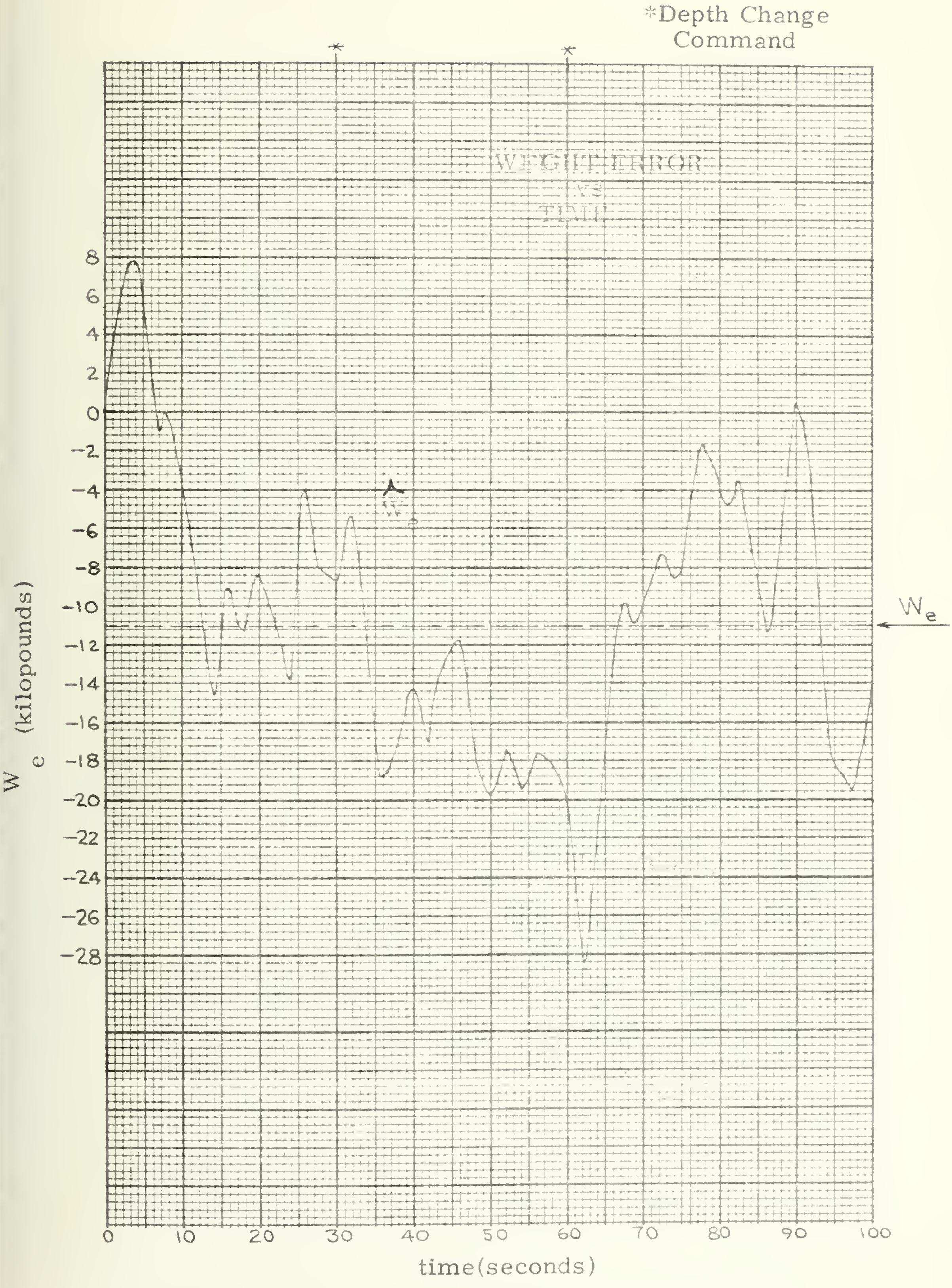

Figure 13. K_{f2} , Step Input, Noise (Table VIII)

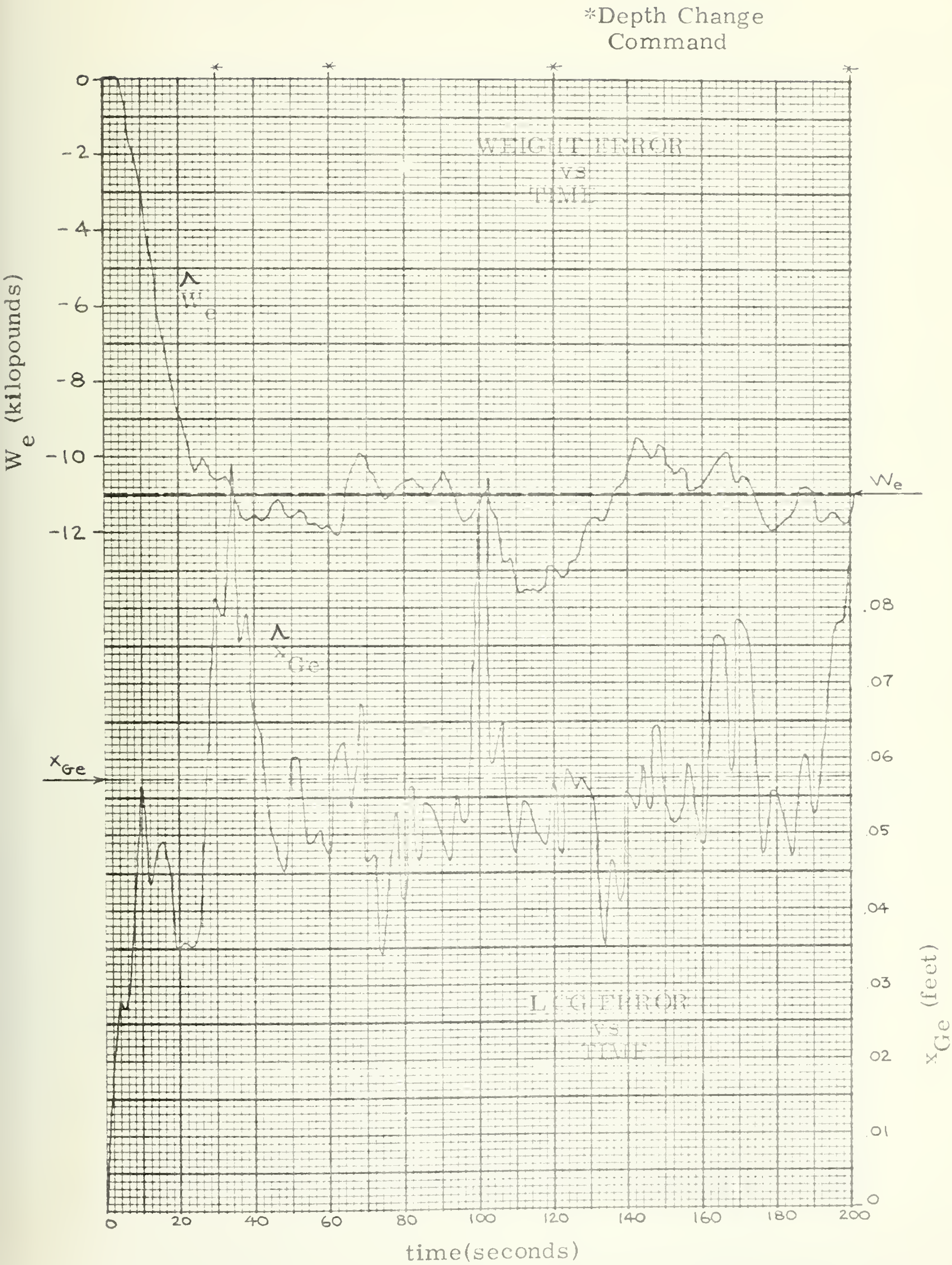

Figure 14. $\underline{K}_{f2}$, Step Input, Noise (Table IX)

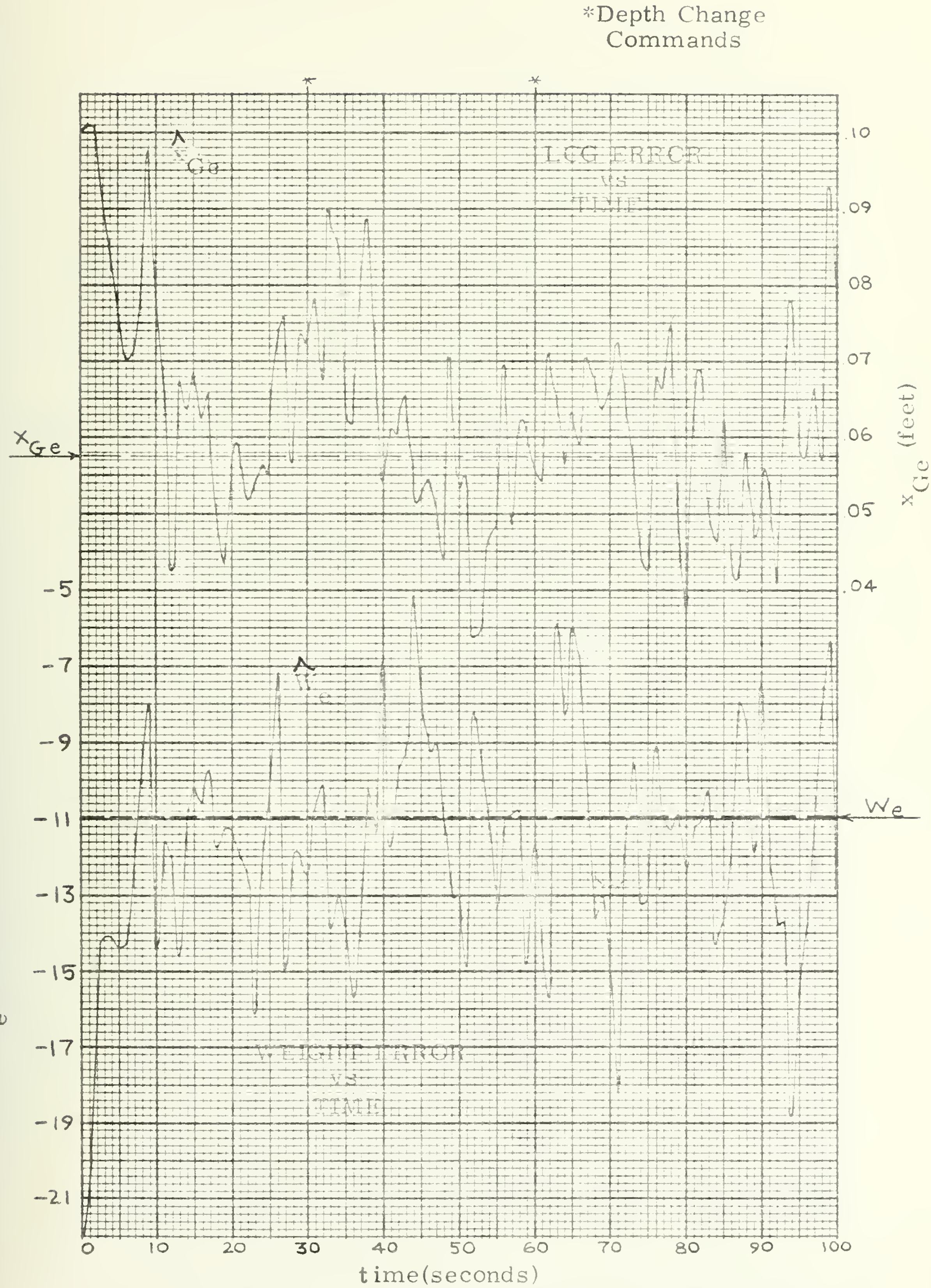

Figure 15. $\underline{K}_{f3}$ and $\underline{K}_{f4}$, Step Input, Noise (Table IX)

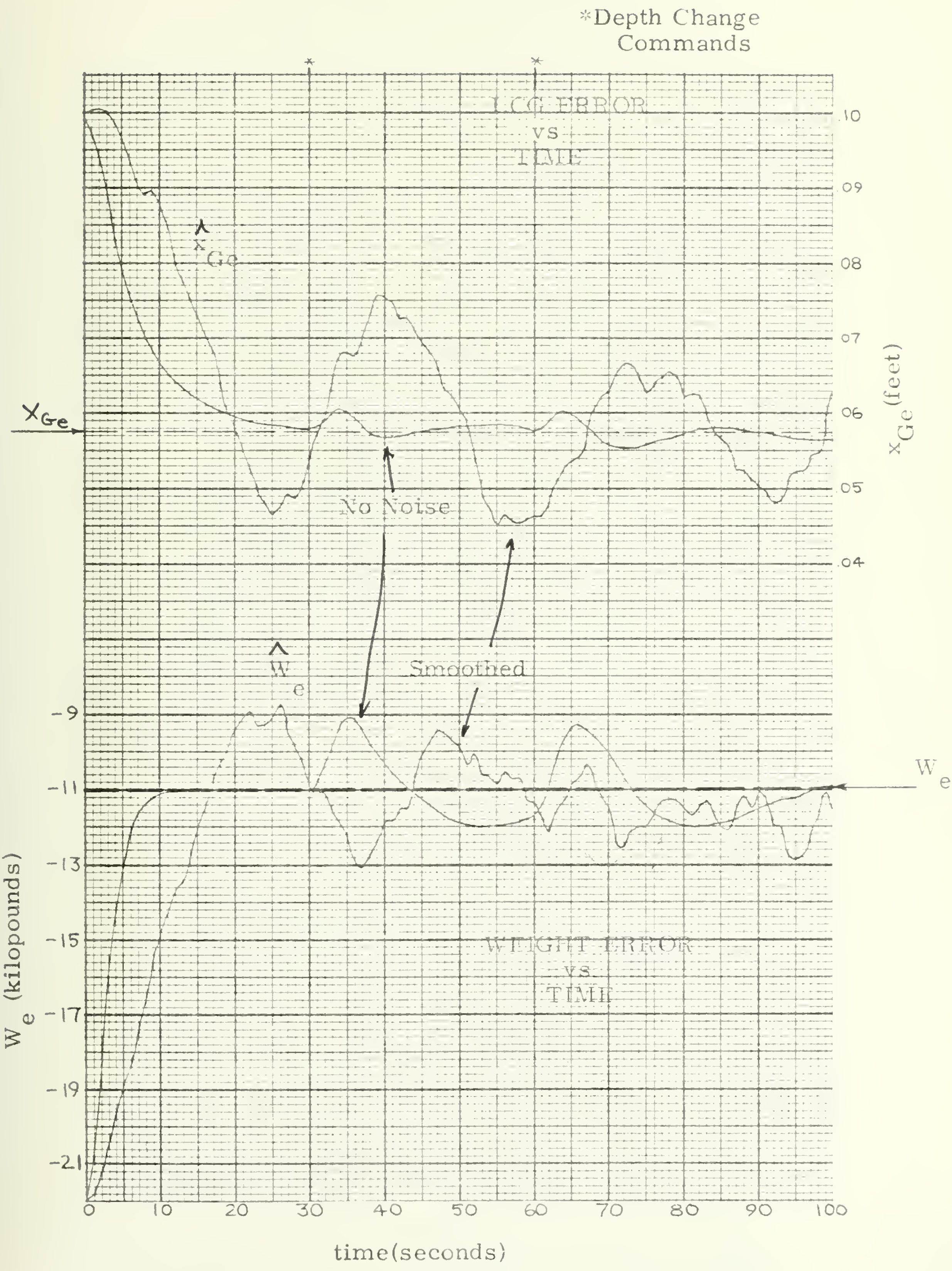

Figure 16. $\underline{K}_{f3}$ and $\underline{K}_{f4}$, Step Input, No Noise and Smoothed Noise (Table IX)

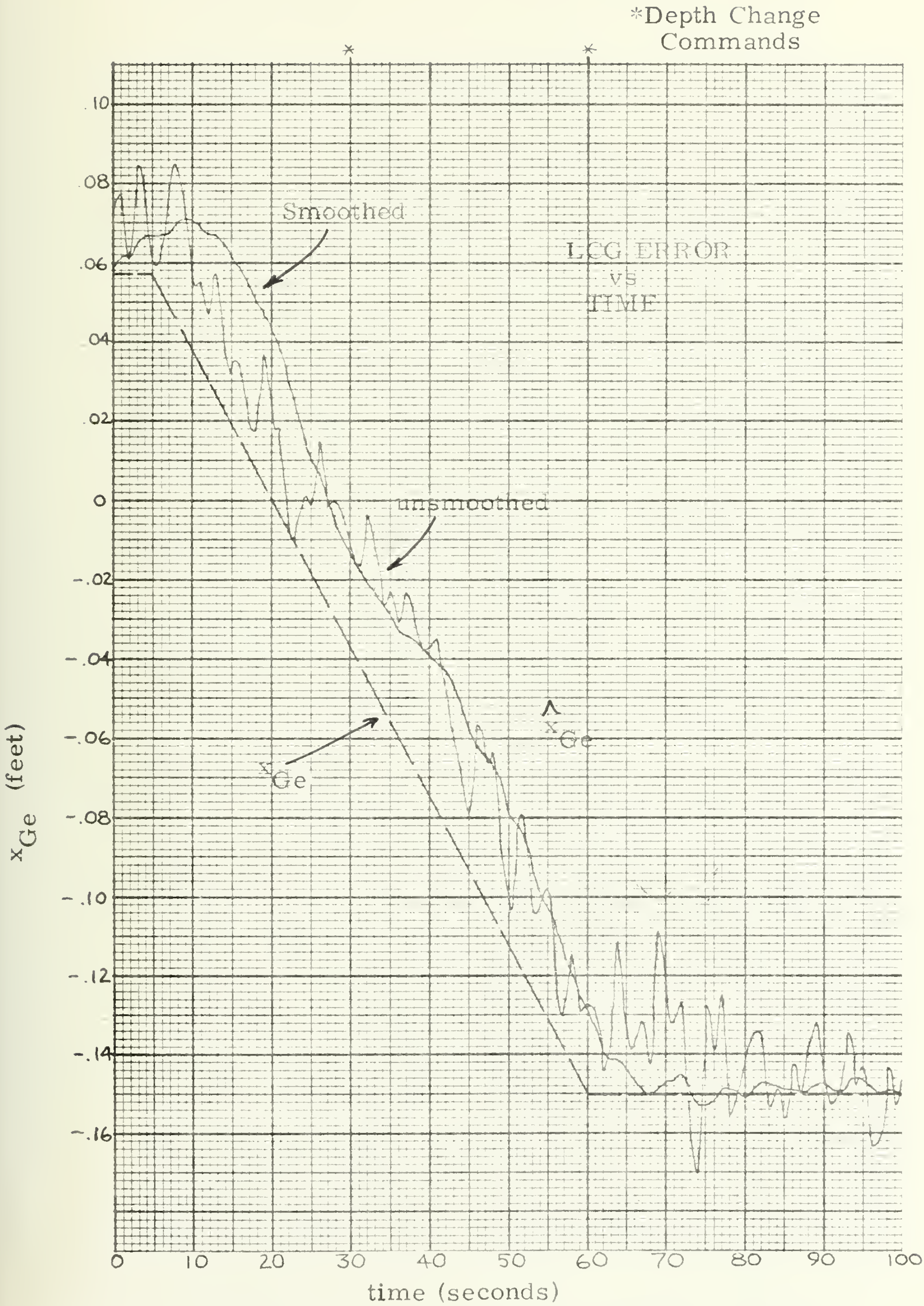

Figure 17. $\underline{K}_{f3}$, Flooding Casualty x_{Ge}, Noise and Smoothed Noise (Table IX)

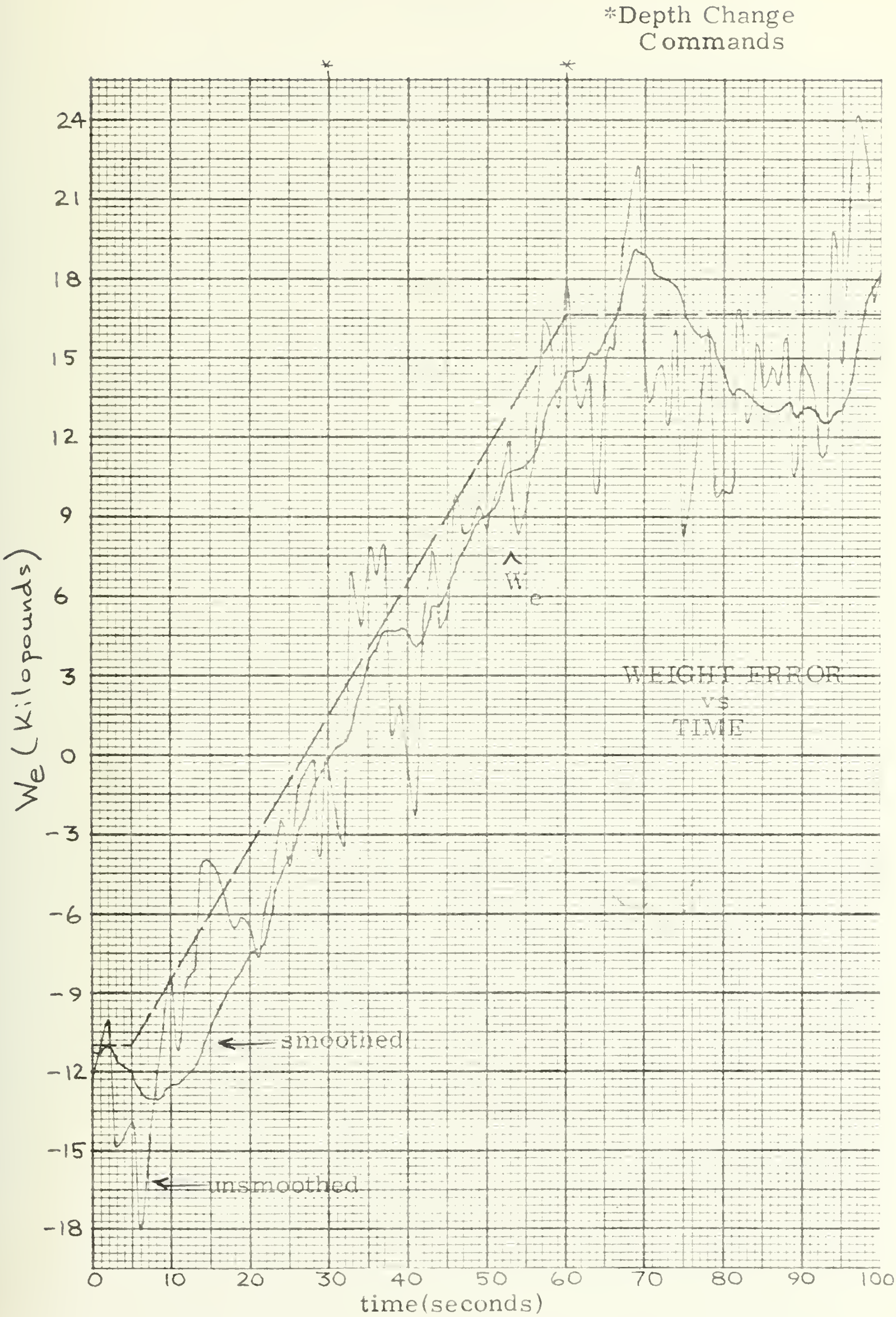

Figure 18. $\underline{K}_{f3}$, Flooding Casualty W_e, Noise and Smoothed Noise (Table IX)

Chapter IV

Conclusions and Recommendations

1. Conclusions

It has been demonstrated that trim analysis lends itself to optimal estimation techniques which manipulate measurements from instruments installed on most U. S. Navy submarines. Sufficient measurements are those of depth, pitch, and pitch rate. Installation of a rate gyro should be a simple task if no instrument, capable of measuring pitch rate, is already installed on a given submarine. Although not required, measurements of rate of descent are useful if available. A use by-product of a Wiener filter designed for this task is its ability to estimate the severity of a flooding casualty. This information could be of benefit to an officer of the deck by alerting him of the casualty and by providing him with information necessary for determining the extent of recovery action required. These estimates can be made while conducting maneuvers in the vertical plane.

A potential problem area exists in the variation of the physical system from the linearized filter model. The simulations described herein are "first cut" in that the filter model coincides with the true system model. In reality, there are two sources of error here. First, the coefficients of motion might not be accurate. Secondly, as deviations from the equilibrium conditions become larger, the accuracy of the linearized equations of the filter is degraded. Motion in the horizontal plane is another deviation for which the filter is not equipped. It can be expected that modeling errors will be reflected by the filter as measurement and estimate errors. Filter instability is another possibility. Being specific about differences in coefficients, some of these variations might be predictable and lend themselves to updating by a computer. Concerning deviations from equilibrium conditions, the control surfaces are probably the worst offenders. However, they are treated as inputs by the filter. Hence, they need not be linear, and it would be satisfactory if forces and moments were stored in a computer as functions of plane deflection. Filter gains would remain the same. The moment of inertia can also change appreciably, depending on how water is distributed in the trim system. This phenomenon is sometimes utilized by submariners to "tune" the resonant frequency of the

submarine in pitch near the surface. It might be advisable, therefore, to feedback water levels in the trim tanks to the filter.

Another potential problem area is that of disturbances. The filter design is based upon the assumption that they are stationary, normally distributed random variables with known statistical properties. While it might well be possible to approach this situation for the measurement distrubances, it is quite another matter for external disturbances. Although no problem exists while the submarine is deep, near surface effects can be very dramatic. In fact, the submarine's response might be very "broad band" compared with typical sea spectrums. It is under such circumstances that the smoother becomes valuable. The best period over which the smoothing should be done might well be the subject of another thesis. However, if the trim error changes very slowly, which it usually does, the luxury of longer periods could be used to obtain smoother estimates.

There are other sources of error which have been neglected such as hull compressibility effects and free surface effects of fluids in tanks or the bilges. It is felt that if not already capable, the filter can be suited to such problems by imaginative uses of the covariance matrices to reflect uncertainties.

Concerning filter gains, susceptibility to noise varies inversely with filter response time. For a true Kalman filter, the gains are functions of time so that initial errors are heavily weighted to ensure rapid convergence followed by relatively smooth tracking. While this is good for variables with large variations, it is not felt to be necessary for trim error variables. This might not be true if the filter is intended to detect and measure flooding casualties. Since these objectives are contradictory, it might be well to consider two separate filters rather than one which does neither job well.

It is not expected that real time applications would be difficult, given the present state-of-the-art of computer technology. Despite the Author's usage of the relatively slow fourth order Runge-Kutta technique, complete problem set-ups and simulations took about half of

the real time simulated on the IBM 360 computer.

2. Recommendations

Before the actual application of such a filter to a test platform, the following factors should be considered:

The filter must be manipulated to be as sophisticated as is useful for the given hardware. This means considering such factors as making the filter discrete or continuous. Shall the gains be calculated on the submarine's computer, or precalculated and stored therein? Should the system matrices be precalculated or stored? Before making such decisions, an extensive sensitivity analysis should be conducted to ascertain the effect of previously described uncertainties on the filter's performance. Most challenging would be the study and simulation of near surface effects.

Reference (7) is packed with practical considerations and, in the opinion of the Author, could be of immense value in any such undertaking. Contained therein are many examples of and inovative solutions to such problems.

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Appendix A

Submarine Equations of Motion in the Vertical Plane

1. General

The first step in the solution of the control and estimation problems is the derivation and linearization of the equations of motion of the submarine. In order to restrain the magnitude of the problem, this derivation is limited to the three degrees of freedom in the vertical plane. These are pitch, heave, and surge. Although this constrains the applicability of the results somewhat, the constraint should not be too severe for two reasons. First, submarines spend the vast majority of their underway time traveling on a straight course between the proverbial points A and B. Secondly, for most course changes, it is expected that the coupling between the vertical and horizontal planes will not be so severe as to render the trim filter useless. In fact, most of the coupling is through second order terms which would be dropped in the linearization process herein. In short, it is not felt that the slight increase in accuracy during relatively rare manuevers justifies the added complexity brought upon by the added three degrees of freedom in the horizontal plane.

The basic forms of the equations are:

$$\underline{F} = d/dt\,(m\,\underline{U}_G) \tag{A-1}$$

and

$$\underline{T}_G = d/dt\,\underline{H}_G \tag{A-2}$$

(A-1) simply says that the rate of change of momentum of a body relative to an inertial reference is equal to the resultant of the applied forces. Likewise, equation (A-2) implies that the rate of change of the angular momentum of a body is equal to the applied torque. Figure 1 on page xx illustrates the axes systems. The x_o and z_o axes are fixed relative to the earth and are directed horizontally and vertically downward respectively. The earth will be assumed to be an inertial reference. Such an assumption will cause negligible errors even for

those relatively high speeds attained by submarines returning from extended deployments.

The x and z axes are the body axes of the submarine. Their origin is the point assumed to be the body axes origin during the derivation of the coefficients of motion. However, due to the changes in the position of the center of gravity relative to the submarine for various trim conditions, the origin of the body axes and the C. G. will rarely coincide. Furthermore, the x-z plane is also a plane of geometric symmetry of the submarine. This is also assumed to be a plane of mass symmetry. Hence, the y axis which is perpendicular to the x-z plane, is assumed to be a principal axis of inertia.

2. Inertial Forces and Moments

Using the techniques described in reference (1), the right hand sides of the basic equations can be broken down into the various components of the inertial reactions relative to the submarine's axes origin. (Wherever possible, the standardized nomenclature of reference (2) has been utilized.) Starting with equation (A-1),

$$d/dt\,(m\underline{U}_G) = \dot{m}\,\underline{U}_G + m\,\dot{\underline{U}}_G \tag{A-3}$$

Normally, rates of change of the mass and the center of gravity of the vehicle are considered negligible for marine applications and are omitted. In this case, they shall be retained for the sake of generality during trimming or flooding of the submarine.

The velocity of the CG becomes

$$\underline{U}_G = \underline{U} + \underline{U}_{rel} + \underline{\Omega} \times \underline{R}_G \tag{A-4}$$

where

$$\underline{U} = u\hat{i} + w\hat{k} \tag{A-5}$$

$$\underline{U}_{rel} = u_{rel}\hat{i} + w_{rel}\hat{k} \tag{A-6}$$

$$\underline{\Omega} = q\hat{j} \tag{A-7}$$

$$\underline{R}_G = x_G\hat{i} + z_G\hat{k} \tag{A-8}$$

Hence,

$$\underline{\Omega} \times \underline{R}_G = z_G q\hat{i} + x_G q\hat{k} \tag{A-9}$$

After appropriate substitutions and grouping of terms, equation (A-4) becomes

$$\underline{U}_G = (u + u_{rel} + z_G q)\hat{i} + (w + w_{rel} - x_G q)\hat{k} \tag{A-10}$$

Next, the acceleration of the CG must be described.

$$\dot{\underline{U}}_G = \dot{\underline{U}} + \dot{\underline{U}}_{rel} + \dot{\underline{\Omega}} \times \underline{R}_G + \underline{\Omega} \times \underline{\Omega} \times \underline{R}_G + 2\,\underline{\Omega} \times \underline{U}_{rel} \tag{A-11}$$

It can be seen in equation (A-11) that there are five components in the acceleration of the CG. The first is that due to the acceleration of the body axes relative to the inertial axes. The second is that due to the translational acceleration of the CG relative to the body axes. The third is due to the angular acceleration of the body axes and the fourth is the centripetal acceleration due to the body axis rotation.

The last contribution is that of the Coriolis acceleration. It is now necessary to reduce the components of $\dot{\underline{U}}_G$ further. Differentiating equation (A-5) with respect to time,

$$\dot{\underline{U}} = \dot{u}\hat{i} + \dot{w}\hat{k} + u\, d\hat{i}/dt + w\, d\hat{k}/dt \tag{A-12}$$

The first two terms on the right hand side of equation (A-12) represent the translational acceleration of the body axes. The second two terms reflect the change in the orientation of the unit vectors due to the rotation of the body axes, ie.,

$$d\hat{i}/dt = \underline{\Omega} \times \hat{i} = -q\hat{k} \tag{A-13}$$

and

$$d\hat{k}/dt = \underline{\Omega} \times \hat{k} = q\hat{i} \tag{A-14}$$

Hence, equation (A-12) becomes

$$\dot{\underline{U}} = (\dot{u} + wq)\hat{i} + (\dot{w} - uq)\hat{k} \tag{A-15}$$

Also,

$$\dot{\underline{U}}_{rel} = \dot{u}_{rel}\hat{i} + \dot{w}_{rel}\hat{k} \tag{A-16}$$

$$\dot{\underline{\Omega}} = \dot{q}\hat{j} \tag{A-17}$$

So,

$$\dot{\underline{\Omega}} \times \underline{R}_G = z_G\dot{q}\hat{i} + x_G q\hat{k} \tag{A-18}$$

Using equation (A-9)

$$\underline{\Omega} \times \underline{\Omega} \times \underline{R}_G = -x_G q^2 \hat{i} - z_G q^2 \hat{k} \tag{A-19}$$

Finally,

$$2\underline{\Omega} \times \underline{U}_{rel} = 2 q w_{rel} \hat{i} - 2 q u_{rel} \hat{k} \tag{A-20}$$

Substituting equations (A-15, 16, 18, 19, and 20) into equation (A-11) and grouping terms yields,

$$\begin{aligned} \dot{\underline{U}}_G = &(\dot{u} + wq + \dot{u}_{rel} + z_G \dot{q} - x_G q^2 + 2 q w_{rel}) \hat{i} \\ &+ (\dot{w} - uq + \dot{w}_{rel} - x_G \dot{q} - z_G q^2 - 2 q u_{rel}) \hat{k} \end{aligned} \tag{A-21}$$

Next equations (A-10 and 21) are substituted into equation (A-3), yielding

$$d/dt\,(m \underline{U}_G) = X_t \hat{i} + Z_t \hat{k} \tag{A-22}$$

where the surge component of the rate of change in linear momentum is

$$\begin{aligned} X_t = &\dot{m}(u + u_{rel} + z_G q) \\ &+ m(\dot{u} + wq + \dot{u}_{rel} + z_G \dot{q} - x_G q^2 + 2 q w_{rel}) \end{aligned} \tag{A-23}$$

and the heave component is

$$\begin{aligned} Z_t = &\dot{m}(w + w_{rel} - x_G q) \\ &+ m(\dot{w} - uq + \dot{w}_{rel} - x_G \dot{q} - z_G q^2 - 2 q u_{rel}) \end{aligned} \tag{A-24}$$

It is necessary to rewrite both sides of equation (A-2) at this stage. Expressing torque about the body axis,

$$\underline{T}_G = \underline{T} - \underline{R}_G \times \underline{F} \tag{A-25}$$

where

$$\underline{R}_G \times \underline{F} = (z_G X_t - x_G Z_t)\,\hat{j} \tag{A-26}$$

On the right hand side of equation (A-2) is

$$\underline{H}_G = \underline{I}_G\,\underline{\Omega} \tag{A-27}$$

Differentiating equation (A-27) with respect to time,

$$d/dt\,\underline{H}_G = \dot{\underline{I}}_G\,\underline{\Omega} + \underline{I}_G\,\dot{\underline{\Omega}} \tag{A-28}$$

or

$$d/dt\,\underline{H}_G = (\dot{I}_{Gy}\,q + I_{Gy}\,\dot{q})\,\hat{j} \tag{A-29}$$

where

$$I_{Gy} = I_y - m\,(x_G^2 + z_G^2) \tag{A-30}$$

and

$$\dot{I}_{Gy} = \dot{I}_y - \dot{m}\,(x_G^2 + z_G^2) - m\,(2\,x_G\,u_{rel} + 2\,z_G\,w_{rel}) \tag{A-31}$$

Combining (A-25, 26, 29, 30, and 31) and rearranging terms yields

$$\underline{I} = [(z_G X_t - x_G Z_t) + \dot{q} I_y + \dot{q} m(x_G^2 + z_G^2) + q \dot{I}_y$$
$$- q\dot{m}(x_G^2 + z_G^2) - qm(2x_G u_{rel} + 2z_G w_{rel})] \hat{j} \qquad \text{(A-32)}$$

Also

$$\underline{I} = M_t \hat{j} \qquad \text{(A-34)}$$

Several cancellations occur when equations (A-23 and 24) are substituted into equation (A-32). Regrouping terms and equating the results to equation (A-34) yields

$$M_t = \dot{q} I_y + q \dot{I}_y + \dot{m}[z_G(u + u_{rel}) - x_G(w + w_{rel})] \qquad \text{(A-35)}$$
$$+ m[z_G(\dot{u} + wq + \dot{u}_{rel}) - x_G(\dot{w} - uq + \dot{w}_{rel})]$$

This completes the modification of the right hand sides of equations (A-1 and 2). Now the left hand sides must be more adequately described.

3. Applied Forces and Moments

Reference (3) gives a comprehensive breakdown of the applied forces and moments. The coefficients are partial derivatives of the hydrodynamic force or moment with respect to the subscripted variable. For example,

$$M_w = \partial M / \partial w$$

For the vertical plane, the surge component of the applied forces is:

$$
\begin{aligned}
X_t = {} & X_{qq}\, q^2 + X_{\dot{u}}\, \dot{u} + X_{wq}\, wq + X_{uu}\, u^2 + X_{ww}\, w^2 \\
& + A_i\, u^2 + B_i\, u u_c + C_i\, u_c^2 - (W - B) \sin \Theta \\
& + X_{ww\eta}\, w^2 (\eta - 1) + (u^2/U^2) \left[X_{\delta b \delta b}\, \delta_b^2 + X_{\delta s \delta s}\, \delta_s^2 \right. \\
& \left. + X_{\delta s \delta s \eta}\, \delta_s^2 (\eta - 1) \right] + \Sigma \text{ Disturbances (surge)}
\end{aligned}
\tag{A-36}
$$

The heave component of the applied forces is:

$$
\begin{aligned}
Z_t = {} & Z_{\dot{q}}\, \dot{q} + Z_{\dot{w}}\, \dot{w} + Z_{w|q|}\, w|q| + Z_{w|w|}\, w|w| + Z_{ww}\, w^2 \\
& + (W - B) \cos \Theta + Z_{w|w|\eta}\, w|w| (\eta - 1) + (u/U) \{ Z_q\, q \\
& + Z_{|q|\delta s} |q| \delta_s + Z_w\, w + Z_{|w|} |w| + [Z_{q\eta}\, q + Z_{w\eta}\, w] \\
& \cdot (\eta - 1) \} + (u^2/U^2) [Z_* + Z_{\delta b}\, \delta_b + Z_{\delta s}\, \delta_s \\
& + Z_{\delta s \eta}\, \delta_s (\eta - 1)] + \Sigma \text{ Disturbances (heave)}
\end{aligned}
\tag{A-37}
$$

For the vertical plane, the only component of the applied torque is the pitch component. The applied torque in pitch is:

$$
\begin{aligned}
M_t = {} & M_{\dot q}\dot q + M_{q|q|}\, q|q| + M_{\dot w}\dot w + M_{|w|q}|w|q \\
& + M_{w|w|}\, w|w| + M_{ww} w^2 - (x_G W - x_B B)\cos\Theta \\
& - (z_G W - z_B B)\sin\Theta + M_{w|w|\eta}\, w|w|(\eta - 1) \\
& + (u/U)\left\{ M_q q + M_{|q|\delta s}|q|\delta_s + M_w w \right. \\
& \left. + M_{|w|}|w| + [M_{q\eta} q + M_{w\eta} w](\eta - 1) \right\} \qquad \text{(A-38)} \\
& + (u^2/U^2)\left[M_* + M_{\delta b}\delta_b + M_{\delta s}\delta_s \right. \\
& \left. + M_{\delta s\eta}\delta_s(\eta - 1) \right] + \sum \text{Disturbances(pitch)}
\end{aligned}
$$

4. Change of Variables

Generally, the motions of a submarine in the vertical plane are described in terms of ahead speed (u), pitch (Θ), and depth (z_o with the origin at the sea surface). These are illustrated in Figure 1 on page xx. This is due in part to the measurements which are available. The preceeding equations of motion must be modified so as to use the available measurements directly. The required change of variables is accomplished using the following equations:

$$\dot{z}_o = w \cos\Theta - u \sin\Theta \tag{A-39}$$

$$\ddot{z}_o = \dot{w} \cos\Theta - wq \sin\Theta - \dot{u} \sin\Theta - uq \cos\Theta \tag{A-40}$$

Furthermore, for the control and estimation aspects of this thesis it is useful to express the variables in terms of command depth (commonly referred to as ordered depth), z_{oc}; deviation from ordered depth, z_{oe}; command pitch (ordered bubble), Θ_c; and deviation from ordered bubble, Θ_e.

Hence,

$$z_o = z_{oc} + z_{oe} \tag{A-41}$$

and

$$\Theta = \Theta_c + \Theta_e \tag{A-42}$$

Since Θ_c and z_{oc} are piecewise constant,

$$\dot{z}_{oe} = \dot{z}_o \tag{A-43}$$

$$\ddot{z}_{oe} = \ddot{z}_o \tag{A-44}$$

$$q = \dot{\Theta}_e \tag{A-45}$$

, and

$$\dot{q} = \ddot{\Theta}_e \tag{A-46}$$

The following additional transformations result:

$$w = \dot{z}_{oe}\left[\cos(\Theta_c + \Theta_e)\right]^{-1} + u \tan(\Theta_c + \Theta_e) \qquad \text{(A-47)}$$

, and

$$\dot{w} = \ddot{z}_{oe}\left[\cos(\Theta_c + \Theta_e)\right]^{-1} + u\,\dot{\Theta}_e + (w\,\dot{\Theta}_e + \dot{u})\tan(\Theta_c + \Theta_e) \qquad \text{(A-48)}$$

5. Linearization

The application of optimal control and estimation theory is well served if the equations of motion are linearized. This is accomplished using the techniques described in reference (4). Basically, a Taylor's series expansion of all variables is made about an equilibrium condition. After substituting these expansions into the equations of motion, terms of higher than first order are omitted. For small perturbations about the equilibrium condition, the actual equations are very closely approximated by the linearized equations. As the magnitude of the pertubations increases, the accuracy of the linearized equations is degraded.

The equilibrium condition used will be for the submarine traveling horizontally at a steady speed and with a steady pitch angle. The object of this thesis is to determine the magnitude of pertubations of the submarine's trim from the "in trim" condition. Therefore, as part of the equilibrium condition it is necessary to include the "in trim" state of weight and longitudinal center of gravity. More details on the equilibrium conditions appear in Appendix B. The linearized variables are as follows:

$$\sin\Theta = \Theta_e \cos\Theta_c + \sin\Theta_c \qquad \text{(A-49)}$$

$$\cos\Theta = \cos\Theta_c - \Theta_e \sin\Theta_c \tag{A-50}$$

Practically speaking, unless the submarine is grossly out of trim, the magnitude of the ordered bubble for depth keeping purposes is always less than seven degrees. Therefore, the second term in equation (A-50) is of the same order of magnitude as a second order term and can be dropped, leaving

$$\cos\Theta = \cos\Theta_c \tag{A-51}$$

Other linearized variables are

$$\tan\Theta = \Theta_e + \tan\Theta_c \tag{A-52}$$

$$u = u_o + \Delta u \tag{A-53}$$

$$\dot{u} = \Delta\dot{u} \tag{A-54}$$

$$u_{rel} = \Delta u_{rel} \tag{A-55}$$

$$W = W_o + W_e \qquad (m = m_o + m_e) \tag{A-56}$$

$$\dot{W} = \Delta\dot{W} \tag{A-57}$$

$$I_y = I_{yo} + \Delta I_y \tag{A-58}$$

$$\dot{I}_y = \Delta\dot{I}_y \tag{A-59}$$

$$x_G = x_{Go} + x_{Ge} \tag{A-60}$$

$$q = \Delta q \tag{A-61}$$

$$\dot{q} = \Delta \dot{q} \tag{A-62}$$

Linearized products of certain variables are:

$$u \tan\Theta = u_o \Theta_e + u_o \tan\Theta_c + \Delta u \tan\Theta_c \tag{A-63}$$

Using equation (A-63) to linearize equation (A-47) yields

$$w = \dot{z}_e (\cos\Theta_c)^{-1} + u_o \Theta_e + u \tan\Theta_c + \Delta u \tan\Theta_c \tag{A-64}$$

which can be used in the linearization of equation (A-48):

$$\dot{w} = \ddot{z}_{oe} (\cos\Theta_c)^{-1} + u_o (1 + \tan^2\Theta_c) \dot{\Theta}_e + \dot{u} \tan\Theta_c \tag{A-65}$$

From equation (A-64) it can be seen that

$$w_o = u_o \tan\Theta_c \tag{A-66}$$

and

$$\Delta w = \dot{z}_{oe} (\cos\Theta_c)^{-1} + u_o \Theta_e + \Delta u \tan\Theta_c \tag{A-67}$$

These variables are used to more clearly illustrate the following linearized variables:

$$|w_o| = u_o\,|\tan\theta_c|^{*} \qquad \text{(A-68)}$$

$$w^2 = w_o^2 + 2\,w_o\,\Delta w \qquad \text{(A-69)}$$

$$w\,|w| = w_o|w_o| + 2|w_o|\,\Delta w \qquad \text{(A-70)}$$

Also,

$$|q| = 0 \qquad \text{(A-71)}$$

$$wq = u_o \tan\theta_c \;\; \dot{\theta}_e \qquad \text{(A-72)}$$

$$u^2 = u_o^2 + 2\,u_o\,\Delta u \qquad \text{(A-73)}$$

$$\delta_b = \Delta\,\delta_b \qquad \text{(A-74)}$$

$$\delta_s = \Delta\,\delta_s \qquad \text{(A-75)}$$

$$\alpha = \theta_c + \Delta\alpha \qquad \text{(A-76)}$$

$$\cos\alpha = \cos\theta_c + \Delta\alpha\,\sin\theta_c \qquad \text{(A-77)}$$

Once again, since θ_c is expected to be less than seven degrees, the second term of equation (A-77) is of the same order of magnitude as a second order term. Therefore it is dropped, leaving

$$\cos\alpha = \cos\theta_c \qquad \text{(A-78)}$$

*For ahead motion only

Since $U = u/\cos\alpha$

the linearized version of velocity becomes

$$U = (u_o + \Delta u)/\cos\theta_c \tag{A-79}$$

Hence,

$$u/U = 1/\cos\theta_c \tag{A-80}$$

For the axial propeller thrust equations

$$\eta = u_c/U \tag{A-81}$$

where

$$u_c = u_{co} + \Delta u_c \tag{A-82}$$

for variable rpm simulations.

Linearizing equation (A-82)

$$\eta = u_{co}/U_o + \Delta u_c/U_o - u_{co}\,\Delta U/U_o^2 \tag{A-83}$$

Substituting the components of equation (A-79) in equation (A-83) yields

$$\eta = u_{co}\cos\theta_c/u_o + \Delta u_c\cos\theta_c/u_o - u_{co}\,\Delta u\cos\theta_c/u_o^2 \tag{A-84}$$

Therefore,

$$(\eta - 1) = (u_{co} \cos\Theta_c / u_o) - 1$$
$$+ (\cos\Theta_c / u_o)\Delta u_c - (u_{co}\cos\Theta_c / u_o^2)\Delta u \qquad \text{(A-85)}$$

For the sake of the simulations herein, rpm is considered to remain constant. This does not affect the trim filter in that it does not attempt to estimate changes in surge. The filter merely uses the measurements of Δu. This policy simplifies equation (A-85):

$$(\eta - 1) = C_{\eta 1} + C_{\eta 2}\Delta u \qquad \text{(A-86)}$$

where

$$C_{\eta 1} = (u_{co}\cos\Theta_c / u_o) - 1 \qquad \text{(A-87)}$$

and

$$C_{\eta 2} = -u_{co}\cos\Theta_c / u_o^2 \qquad \text{(A-88)}$$

This completes the bulk of the linearization of the variables with the exceptions of $\dot{w}_{rel}$, $\dot{u}_{rel}$, w_{rel}, and z_G. For the linearized equations, it shall be assumed that changes in the height of the CG are extremely small for all practical cases. Hence, the effect on the dynamics of the submarine as predicted by the linearized equations would be negligible. From the assumption that

$$\Delta z_G = 0 \qquad \text{(A-89)}$$

it follows also that

$$w_{rel} = \dot{w}_{rel} = 0 \qquad \text{(A90)}$$

for the linearized equations.

In the longitudinal direction, there exists the possibilities of much larger moment arms for flooding and trimming water. Hence Δx_G is certainly a variable to reckon with. The rate of change of Δx_G (u_{rel}) can be expected to be small for typical rates of flooding or trimming. Due to coupling with other variables, u_{rel} does not emerge from the linearization process as a first order term, so one does not have to make a practical decision as to whether or not it should be retained in the linearized equations. This is not the case for $\dot{u}_{rel}$; so the assumption is made, in light of the fact that u_{rel} is so small, that

$$\dot{u}_{rel} = 0 \tag{A-91}$$

for the linearized equations.

Substituting the linearized variables into equations (A-23) and (A-36) and arranging terms yields the linearized surge equation:

$$\begin{aligned} m_o z_G \ddot{\theta}_e &+ (m_o - X_{\dot{u}})\dot{u} + (u_o/g)\dot{W} \\ &= (X_{\Delta w}/\cos\theta_c)\dot{z}_{oe} + X_{\Delta\theta e}\,\theta_e + X_{\Delta\dot{\theta} e}\,\dot{\theta}_e \\ &+ X_{\Delta u}\,\Delta u - \sin\theta_c\, W_e + F_{xs} + F_{x\,ext} \end{aligned} \tag{A-92}$$

where

$$X_{\Delta\theta e} = u_o X_{\Delta w} - (W_o - B)\cos\theta_c \tag{A-93}$$

$$X_{\Delta w} = 2 w_o (X_{ww} + C_{\eta 1} X_{ww\eta}) \tag{A-94}$$

$$w_o = u_o \tan\theta_c \tag{A-95}$$

$$X_{\Delta\dot{\theta}e} = w_o\left(X_{wq} - m_o\right) \tag{A-96}$$

$$X_{\Delta u} = 2u_o\left(A_i + X_{uu}\right) + B_i u_c + C_{\eta 2} X_{ww\eta} w_o^2 + X_{\Delta w} \tan\Theta_c \tag{A-97}$$

$$F_{Xs} = w_o^2\left(X_{ww} + C_{\eta 1} X_{ww\eta}\right) + C_i u_c^2 + u_o\left[u_o\left(A_i + X_{uu}\right) + B_i u_c\right] - \left(W_o - B\right)\sin\Theta_c \tag{A-98}$$

and $C_{\eta 1}$ and $C_{\eta 2}$ are defined by equations (A-87) and (A-88).

Equations (A-24) and (A-37) lead to the linearized heave equation:

$$\begin{aligned} Z_{\Delta\ddot{z}oe}\ddot{z}_{oe} &+ Z_{\Delta\ddot{\theta}e}\ddot{\Theta}_e + Z_{\Delta\dot{w}}\tan\Theta_c\,\dot{u} \\ &+ (w_o/g)\dot{W} = \left(Z_{\Delta w}/\cos\Theta_c\right)\dot{z}_{oe} + Z_{\Delta w} u_o \Theta_e \\ &+ Z_{\Delta\dot{\theta}e}\dot{\Theta}_e + \cos^2\Theta_c Z_{\delta b}\,\delta_b + Z_{\Delta\delta s}\,\delta_s \\ &+ Z_{\Delta u}\,\Delta u + \cos\Theta_c W_e + F_{Zs} + F_{Zext} \end{aligned} \tag{A-99}$$

where

$$Z_{\Delta\ddot{z}oe} = Z_{\Delta\dot{w}}/\cos\Theta_c \tag{A-100}$$

$$Z_{\Delta\dot{w}} = m_o - Z_{\dot{w}} \tag{A-101}$$

$$Z_{\Delta\ddot{\theta}e} = -x_{Go}\, m_o - Z_{\dot{q}} \tag{A-102}$$

$$Z_{\Delta w} = 2|w_o|\left(Z_{w|w|} + C_{\eta 1} Z_{w|w|\eta}\right) + 2 Z_{ww} w_o + \cos\theta_c \left(Z_w + C_{\eta 1} Z_{w\eta}\right) \tag{A-103}$$

$$Z_{\Delta u} = w_o C_{\eta 2}\left(Z_{w|w|\eta}|w_o| + Z_{w\eta}\cos\theta_c\right) + Z_{\Delta w}\tan\theta_c \tag{A-104}$$

$$Z_{\Delta\delta s} = \left(Z_{\delta s} + C_{\eta 1} Z_{\delta s\eta}\right)\cos^2\theta_c \tag{A-105}$$

$$F_{zs} = \left(W_o - B + Z_{|w|}|w_o| + Z_* \cos\theta_c\right)\cos\theta_c + w_o\left[|w_o|\left(Z_{w|w|} + C_{\eta 1} Z_{w|w|\eta}\right) + w_o Z_{ww} + \left(Z_w + C_{\eta 1} Z_{w\eta}\right)\cos\theta_c\right] \tag{A-106}$$

$$Z_{\Delta\dot{\theta}e} = \left(Z_q + C_{\eta 1} Z_{q\eta}\right)\cos\theta_c + u_o\left[m_o - Z_{\Delta\dot{w}}\left(1 + \tan^2\theta_c\right)\right] \tag{A-107}$$

and $|w_o|$, w_o, $C_{\eta 1}$, and $C_{\eta 2}$ are defined in equations (A-68), (A-95), (A-87), and (A-88) respectively.

Finally, equations (A-35) and (A-38) lead to the linearized pitch equation:

$$(M_{\Delta\dot{w}}/\cos\Theta_c)\,\ddot{z}_{oe} + (I_{yo} - M_{\dot{q}})\,\ddot{\Theta}_e + M_{\Delta\dot{u}}\,\dot{u}$$
$$+ M_{\Delta\dot{m}g}\,\dot{W} = (M_{\Delta w}/\cos\Theta_c)\,\dot{z}_{oe} + M_{\Delta\Theta e}\Theta_e \qquad \text{(A-108)}$$
$$+ M_{\Delta\dot{\Theta}e}\,\dot{\Theta}_e + M_{\delta b}\cos^2\Theta_c\,\delta_b + M_{\Delta\delta s}\,\delta_s + M_{\Delta u}\,\Delta u$$
$$- W_o\cos\Theta_c\ x_{Ge} + M_{\Delta mg}W_e + F_{Ms} + F_{Mext}$$

where

$$M_{\Delta\dot{w}} = -x_{Go}m_o - M_{\dot{w}} \qquad \text{(A-109)}$$

$$M_{\Delta w} = 2|w_o|(M_{w|w|} + C_{\eta 1}M_{w|w|\eta}) + 2M_{ww}w_o$$
$$+ (M_w + C_{\eta 1}M_{w\eta})\cos\Theta_c \qquad \text{(A-110)}$$

$$M_{\Delta\dot{u}} = z_G m_o + M_{\Delta\dot{w}}\tan\Theta_c \qquad \text{(A-111)}$$

$$M_{\Delta\dot{m}g} = (z_G u_o - x_{Go}w_o)/g \qquad \text{(A-112)}$$

$$M_{\Delta\Theta e} = u_o M_{\Delta w} - (z_G W_o - z_B B)\cos\Theta_c \qquad \text{(A-113)}$$

$$M_{\Delta\dot{\theta}e} = m_o(-x_{Go}u_o - z_G w_o) + M_{|w|q}|w_o| \quad \text{(A-114)}$$

$$-u_o M_{\Delta\dot{w}}(1 + \tan^2\theta_c) + (M_q + C_{\eta 1} M_{q\eta})\cos\theta_c$$

$$M_{\Delta\delta s} = (M_{\delta s} + C_{\eta 1} M_{\delta s\eta})\cos^2\theta_c \quad \text{(A-115)}$$

$$M_{\Delta u} = M_{\Delta w}\tan\theta_c$$

$$+ w_o C_{\eta 2}(M_{w|w|\eta}|w_o| + M_{w\eta}\cos\theta_c) \quad \text{(A-116)}$$

$$M_{\Delta mg} = -x_{Go}\cos\theta_c - z_G\sin\theta_c \quad \text{(A-117)}$$

$$F_{Ms} = -(z_G W_o - z_B B)\sin\theta_c + w_o\Big[|w_o|(M_{w|w|}$$

$$+ C_{\eta 1} M_{w|w|\eta}) + w_o M_{ww} + (M_w + C_{\eta 1} M_{w\eta})\cos\theta_c\Big]$$

$$+(x_B B - x_{Go} W_o + M_{|w|}|w_o| + M_* \cos\theta_c) \quad \text{(A-118)}$$

$$\cdot \cos\theta_c$$

and $|w_o|$, w_o, $C_{\eta 1}$ and $C_{\eta 2}$ are defined in equations (A-68), (A-95), (A-87), and (A-88) respectively.

Appendix B

Equilibrium Conditions

and Deviations Therefrom

1. General

In the linearization of the euqations of motion, equilibrium conditions were referred to. These are the values about which variables are expanded in the linearization process. The manner of and the logic behind the choice of these conditions is discussed in this appendix. It should be understood that, unless stated otherwise herein, the equilibrium value of any variable is zero. It can be seen that all conditions evolve from the choice of ordered bubble and shaft rpm.

2. Equilibrium Speeds

In the actual operation of a submarine, equilibrium speeds during level flight are functions of equilibrium pitch and shaft rpm. For the computer simulations, shaft rpm was considered to remain constant to avoid the ambiguities which would have arisen in attempting to model the dynamics of the propulsion plant. The filter gains are predicated upon a value of u_c which is a function of shaft rpm. In the actual implementation of the filter, a provision for the input of shaft rpm should be sufficient to keep the filter updated for changes in u_c.

The evaluation of U_o is necessarily one of the first steps in the simulations. The coefficients for the equations of motion are in dimensional form. To calculate them, it is necessary to know the value of U_o.

The equilibrium speed problem is stated thusly:

Given: u_c and Θ_c

Find: u_o, w_o, and U_o

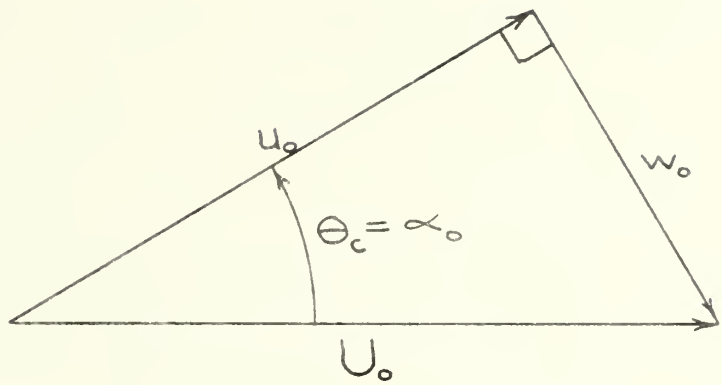

Figure b-1. Representation of Equilibrium Speeds

Figure b-1 illustrates the relationships between most of the variables. They are:

$$U_o = u_o / \cos \Theta_c \tag{B-1}$$

and

$$w_o = u_o \tan \Theta_c \tag{B-2}$$

The solution of the speeds is begun with that of u_o. This is done via the linearized surge equation, equation (A-92). In the equilibrium condition, assuming negligible external disturbances in surge, equation (A-92) reduces to

$$0 = F_{XS} \tag{B-3}$$

which can be rearranged to read as follows:

$$0 = (X_{uu} + A_i)\,u_o^2 + B_i\,u_c u_o + C_i\,u_c^2$$
$$+ [X_{ww} + (\eta_o - 1)\,X_{ww\eta}]\,w_o^2 - (W_o - B)\sin\theta_c \qquad \text{(B-4)}$$

where

$$\eta_o = u_c / U_o = u_c \cos\theta_c / u_o \qquad \text{(B-5)}$$

Equation (B-4) is non-dimensionalized by dividing by $1/2\,\rho\ell^2$.

$$0 = (X'_{uu} + a_i)\,u_o^2 + b_i\,u_c u_o + c_i\,u_c^2$$
$$+ [X'_{ww} + (\eta_o - 1)\,X'_{ww\eta}]\,w_o^2 \qquad \text{(B-6)}$$
$$- (W_o - B)\sin\theta_c / \tfrac{1}{2}\rho\ell^2$$

The solution of the "intrim" weight (W_o) follows (equation B-14). As is illustrated in figure 6, W_o is a function of velocity squared. Capitalizing upon this and the linear relationships between the velocities, it is possible to turn equation (B-6) into something resembling a quadratic equation for u_o:

$$u_o^2 \{X'_{uu} + a_i + (Z'_{|w|}\,|\tan\theta_c| + Z'_{*})\tan\theta_c$$
$$+ [(Z'_{w|w|} + C_{\eta 1} Z'_{w|w|\eta})\,|\tan\theta_c| + Z'_{ww}\tan\theta_c$$
$$+ X'_{ww} + Z'_{w} + C_{\eta 1} Z'_{w\eta}]\tan^2\theta_c\} + b_i\,u_c u_o \qquad \text{(B-7)}$$
$$+ c_i\,u_c^2 + E(F_{Zext})\tan\theta_c / \tfrac{1}{2}\rho\ell^2 = 0$$

where C_{η_1} is defined by equation (A-87). Due to C_{η_1}, equation (B-7) is not actually quadratic, but it does lend itself to a iterative solution. Fortuitously, for the coefficients used in the numerical example herein, equation (B-7) actually is a quadratic equation:

$$A' u_o^2 + B' u_o + C' = 0 \tag{B-8}$$

where

$$A' = X'_{uu} + a_i + (Z'_w + Z'_{ww} \tan\Theta_c + X'_{ww}) \tan^2\Theta_c$$
$$+ (Z'_{|w|} \tan\Theta_c + Z_{w|w|} \tan^2\Theta_c) |\tan\Theta_c| \tag{B-9}$$
$$+ Z'_* \tan\Theta_c$$

$$B' = b_i u_c \tag{B-10}$$

and

$$C' = c_i u_c^2 + E(F_{z\,ext}) \tan\Theta_c / \tfrac{1}{2} \rho \ell^2 \tag{B-11}$$

There are two solutions to equation (B-8), only one of which is realistic. This turns out to be

$$u_o = \frac{-B' - \sqrt{B'^2 - 4A'C'}}{2A'} \tag{B-12}$$

for the coefficients in the numerical example.

Having solved for u_o, equations (B-1) and (B-2) are used to determine U_o and w_o.

3. "In Trim" Condition

The equilibrium trim problem can be stated as follows:

Given: u_c and Θ_c

Find: W_o and x_{Go}

This is accomplished via the linearized heave and pitch equations, equations (A-99) and (A-108) respectively. In equilibrium, the heave equation simplifies to

$$0 = F_{zs} + E(F_{z\,ext}) \qquad \text{(B-13)}$$

This equation is manipulated to find W_o:

$$\begin{aligned} W_o = B - Z_{|w|}|w_o| - Z_* \cos\Theta_c - E(F_{z\,ext})/\cos\Theta_c \\ -\Big[|w_o|\left(Z_{w|w|} + C_{\eta 1} Z_{w|w|\eta}\right) + w_o Z_{ww} \qquad \text{(B-14)} \\ + \left(Z_w + C_{\eta 1} Z_{w\eta}\right)\cos\Theta_c\Big] w_o/\cos\Theta_c \end{aligned}$$

It should be noted that the bouyant force, B, is assumed to remain constant and is equal to the weight of the water displaced by the volume of the entire submarine, including the free flooding volume. Therefore, all changes in the trim of the submarine will be due to changes in the

density of the fluid in which the submarine is submerged, or due to changes of the mass within the envelope formed by the outer skin of the submarine.

Having determined W_o, the linearized pitch equation is used to determine the longitudinal position of the CG for the "in trim" condition. In equilibrium, equation (A-108) simplifies to

$$0 = F_{MS} + E(F_{Mext}) \tag{B-15}$$

Rearranging this equation to solve for x_{Go},

$$\begin{aligned} x_{Go} = \frac{1}{W_o} \Big\{ & x_B B + M_{|w|} |w_o| + M_* \cos\Theta_c \\ & - (z_G W_o - z_B B) \tan\Theta_c + E(F_{Mext})/\cos\Theta_c \\ & + \frac{w_o}{\cos\Theta_c} \big[|w_o| (M_{w|w|} + C_{\eta 1} M_{w|w|\eta}) + w_o M_{ww} \\ & + (M_w + C_{\eta 1} M_{w\eta}) \cos\Theta_c \big] \Big\} \end{aligned} \tag{B-16}$$

4. <u>Deviations from "in Trim Condition</u>

The following equations are repeated for clarity.

$$W = W_o + W_e \tag{A-56}$$

$$x_G = x_{Go} + x_{Ge} \tag{A-60}$$

The means of calculating W_o and x_{Go} are equations (B-14) and (B-16). For the simulations, the submarine was put in an "out of trim"

state by adding weights at various longitudinal locations. Hence,

$$W_e = \sum_{i=1}^{\#\,\text{added}} W_i \tag{B-17}$$

and

$$x_{Ge} = \frac{1}{W} \cdot \sum_{i=1}^{\#\,\text{added}} x_{Gi} W_i \tag{B-18}$$

The purpose of the trim filter is to make a quantitative estimate of these variables.

5. Changes in Trim while Flooding & Pumping

Most of the simulations were conducted for step inputs in trim errors. A more realistic case is that of flooding or pumping. The following equations describe the effect of flooding at a longitudinal location designated x_f. The rate of change of weight error is

$$\dot{W}_e = \dot{W} = \dot{W}_f \tag{B-19}$$

At any instant, the CG is located at

$$x_G = \frac{x_{G\,int} W_{int} + x_f W_f}{W_{int} + W_f} \tag{B-20}$$

Differentiating equation (B-20) with respect to time yields

$$u_{rel} = \dot{W}_f \left(x_f - x_G \right) / W \tag{B-21}$$

where

$$W = W_{int} + \int_{t_1}^{t_2} \dot{W}_f \, dt \tag{B-22}$$

and

$$x_G = x_{Gint} + \int_{t_1}^{t_2} u_{rel} \, dt \tag{B-23}$$

Pumping is simulated by making $\dot{W}_f$ less than zero.

Appendix C

Controller Design

The general problem of the controller design was stated in section 3 of chapter II. The linear system is described by

$$d/dt\ \underline{x}_c = \underline{A}_c\ \underline{x}_c + \underline{B}_c\ \underline{u}_c \tag{C-1}$$

and the purpose of the controller is to minimize the performance index

$$PI = \int_0^T (\underline{x}_c^T\ \underline{Q}_c\ \underline{x}_c + \underline{u}_c^T\ \underline{P}_c\ \underline{u}_c)dt \tag{C-2}$$

where both $\underline{Q}_c$ and $\underline{P}_c$ are symmetric, $\underline{Q}_c$ is positive semi-definite, and $\underline{P}_c$ is positive definite. Physically this means that not all state variables need be controlled and that all control variables must be weighted to preclude an infinite control effort. The optimal control is the feedback of the state:

$$\underline{u}_c = -\underline{K}_c \underline{x}_c \tag{C-3}$$

where

$$\underline{K}_c = \underline{P}_c^{-1}\ \underline{B}_c^T\ \underline{R} \tag{C-4}$$

and $\underline{R}$ is the solution to the matrix Riccati equation (Appendix E).

Concerning the weighting matrices, it is not their absolute magnitudes that matter so much as their relative magnitude which determines the magnitude of the control effort which is proportional to $\underline{Q}_c$ and inversely proportional to $\underline{P}_c$. Also, the relative magnitude of the individual elements within each matrix determines the relative importance of each of the variables. When the steady state solution of the Riccati equation is used to calculate optimal feedback gains, the controller is classified as a linear regulator.

Appendix D

Filter Design

1. Optimal Filter

The filter problem was set forth in section 4 of chapter II. The filter equation is

$$d/dt\ \dot{\hat{\underline{x}}}_f = \underline{A}_f \hat{\underline{x}}_f + \underline{B}_f\ \underline{u}_f + \underline{K}_f(\underline{z}_f - \hat{\underline{y}}_f) \tag{D-1}$$

where

$$\underline{K}_f = \underline{R}\ \underline{C}^T\ \underline{P}_f^{-1} \tag{D-2}$$

and $\underline{R}$ is the solution to the matrix Riccati equation (Appendix E) which minimizes the performance index

$$PI = \int_o^T (\underline{x}_f\ \underline{B}_f\ \underline{Q}_f\ \underline{B}_f^T \underline{x}_f + \underline{u}_f^T\ \underline{P}_f\ \underline{u}_f)dt \tag{D-3}$$

Reference (7) gives excellent descriptions of the physical significance of the solution. The weighting matrices for the filter problem are not so arbitrary as those for the controller problem. For example, $\underline{Q}_f$ is the covariance matrix of the random system disturbances, $\underline{w}$.

$$\underline{Q}_f = E(\underline{w}\,\underline{w}^T) \tag{D-4}$$

For a two dimensional matrix

$$\underline{Q}_f = \begin{bmatrix} E(w_1^2) & E(w_1 w_2) \\ E(w_1 w_2) & E(w_2^2) \end{bmatrix} \tag{D-5}$$

which illustrates the diagonal elements of $\underline{Q}_f$ are the mean square values of the distrubances and the off-diagonal elements are indications of the cross correlation between the elements of $\underline{w}$.

Similarly, for the measurement noise,

$$\underline{P}_f = E(\underline{v}\,\underline{v}^T) \tag{D-6}$$

Physically, the solution of the Riccati equation is the error covariance matrix for the filter,

$$\underline{R} = E(\underline{\tilde{x}}\,\underline{\tilde{x}}^T) \tag{D-7}$$

The Riccati equation for the filter problem is

$$\dot{\underline{R}} = \underline{A}_f \underline{R} + \underline{R}\underline{A}^T - \underline{R}\underline{C}_f^T \underline{P}_f^{-1} \underline{C}_f \underline{R} + \underline{B}_f \underline{Q}_f \underline{B}_f^T \tag{D-8}$$

The first two terms on the right hand side of equation (D-8) result from the unforced system characteristics in the absense of measurements. $\underline{B}_f \underline{Q}_f \underline{B}_f^T$ accounts for the increase in uncertainty due to system noise and $- \underline{R}\,\underline{C}_f^T \underline{P}_f^{-1} \underline{C}_f \underline{R}$ accounts for the decrease in uncertainty due to measurements and their quality.

Looking back at equation (D-2) it can be seen that the filter feedback gains are proportional to the uncertainty in the estimate and inversely proportional to the measurement noise.

One other note: when the steady state solution to the Riccati equation is used to determine filter gains, the filter is classified as a Wiener filter.

2. Sub-Optimal Filter

When the Riccati equation cannot be solved, some other means of determining filter feedback gains must be used. Looking at the filter equation in the absense of noise,

$$d/dt\ \hat{\underline{x}} = A \hat{\underline{x}} + \underline{B}\,\underline{u} - \underline{K}\,\underline{C}\,\tilde{\underline{x}} \tag{D-9}$$

and

$$\tilde{\underline{x}} = \hat{\underline{x}} - \underline{x} \tag{D-10}$$

is the estimate error.

The system equation is

$$d/dt\ \underline{x} = \underline{A}\ \underline{x} + \underline{B}\ \underline{u} \qquad \text{(D-11)}$$

So if equation (D-11) is subtracted from (D-9), the result is

$$d/dt\ \underline{\tilde{x}} = (\underline{A} - \underline{K}\ \underline{C})\ \underline{\tilde{x}} \qquad \text{(D-12)}$$

which shows that the filter will be stable provided $\underline{\tilde{x}}$ goes to zero as time increases. Stated in another manner, the filter will be stable, provided the roots of the filter's characteristic equation

$$\det(\lambda\ \underline{I} - \underline{A} + \underline{K}\ \underline{C}) = 0 \qquad \text{(D-13)}$$

have negative real parts.

As mentioned earlier, it was necessary to select only the fifth and sixth rows of the gains matrix this way, the other gains having been found by Potter's method (Appendix E). It was possible to logically determine the signs of the gains based upon the qualitative behavior of the filter as described by equation (D-9). For example, if z_{oe} and/or $\dot{z}_{oe}$ were less than estimated, implying

$$\tilde{z}_{oe} > 0 \qquad \text{and/or} \qquad \tilde{\dot{z}}_{oe} > 0$$

it would imply that the weight error (W_e) was less than previously estimated. To correct this error, it would be necessary decrease $\hat{W}_e$, so

$$\hat{W} \uparrow \quad \text{if} \quad k_{61} > 0 \quad \text{and} \quad k_{62} > 0$$

Due to the coupling between pitch and heave, the same phenomenon might be due to pitch being greater than estimated which in turn was

due to x_{Ge} being further aft than estimated, and

$$x_{Ge} \downarrow \quad \text{if} \quad k_{51} > 0 \quad \text{and} \quad k_{52} > 0$$

Now if Θ_e and/or $\dot{\Theta}_e$ were less than estimated, ie.,

$$\tilde{\Theta}_e > 0 \quad \text{and/or} \quad \tilde{\dot{\Theta}} > 0$$

it might be due to x_{Ge} being further forward than estimated and

$$x_{Ge} \uparrow \quad \text{if} \quad k_{53} < 0 \quad \text{and} \quad k_{54} < 0$$

Nothing conclusive can be said about W_e in this case unless x_{Ge} is known, so both k_{63} and k_{64} were set equal to zero. These results are summarized in Table d-1.

$k_{51} > 0$	$k_{52} > 0$	$k_{53} < 0$	$k_{54} < 0$
$k_{61} > 0$	$k_{62} > 0$	$k_{63} = 0$	$k_{64} = 0$

Table d-1. Signs of Unknown Gains

Using the constraints in the table, gains were selected at random and the eigenvalues of equation (D-13) were calculated. Then the gains were varied individually and in pairs to determine the sensitivity of the filter's poles (eigenvalues) to each of the gains. In this manner, a sub-optimum was selected.

It might be added that the decision concerning k_{63} and k_{64} being zero was adhered to after discovering that comparable gains of either sign had very little effect on the poles of the filter.

3. Smoothing

The filtered estimates of W_e and x_{Ge} tended to reflect the noise in the system and measurements. The estimates were smoothed using the least squares algorithm in reference (9).

The "best fit" curve for a number of points is assumed to have the form

$$x = mt + b \tag{D-14}$$

It is the best curve in that it minimizes the squares of the deviations,

$$d_i = x_i - (mt_i + b) \tag{D-15}$$

, of the data points from the line. The line will pass through the centroid of the n data points,

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{D-16}$$

and

$$\bar{t} = \frac{1}{n} \sum_{i=1}^{n} t_i \tag{D-17}$$

The slope of the line will be

$$m = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(t_i - \bar{t})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \tag{D-18}$$

so that equation (D-14) becomes

$$x = m(t - \bar{t}) + \bar{x}$$

and the standard deviation of the data about the line is

$$\bar{d} = \left\{ \frac{\left[\sum_{i=1}^{n} (x_i - \bar{x})^2 \sum_{i=1}^{n} (t_i - \bar{t})\right] - \left[\sum_{i=1}^{n} (x_i - \bar{x})(t_i - \bar{t})\right]^2}{n \sum_{i=1}^{n} (t_i - \bar{t})^2} \right\}^{1/2} \tag{D-19}$$

This algorithm was implemented in a subroutine named LESQR (Appendix F).

This could be used for the trim analysis technique mentioned in chapter I. The plane deflections and pitch angle could be smoothed while the submarine stayed in the vicinity of ordered depth. Presumably slopes would be zero and the results would merely be the averaged values of the angles which could be substituted into the linearized heave and pitch equations (equations A-99 and A-108). These would then be solved algebraically to determine the state of the trim. The simplified equations are

$$\begin{aligned} 0 = {} & Z_{\Delta w} u_o \theta_e + \cos^2\theta_c Z_{\delta b} \delta_b \\ & + Z_{\Delta \delta s} \delta_s + \cos\theta_c W_e \end{aligned} \tag{D-20}$$

and

$$\begin{aligned} 0 = {} & M_{\Delta \theta e} \theta_e + M_{\delta b} \cos^2\theta_c \delta_b \\ & - W_o \cos\theta_c x_{Ge} + M_{\Delta mg} W_e \end{aligned} \tag{D-21}$$

Of course, W_o is based upon knowledge of the mean external disturbance in heave (equation B-14). If this is not known, the value of W_o for no mean heave distrubance could be used without serious effects. W_e could be driven to zero based upon equation (D-20) without knowledge of external disturbances. Then equation (D-21) would indicate the sign of x_{Ge} and closely approximate its magnitude. The true solution would be converged upon as the submarine was trimmed.

Appendix E

The Matrix Riccati Equation

1. General

Calculation of the feedback gains for the optimal controller and the Kalman filter requires the solution of the matrix Riccati equation. The equation is the solution to the following problem.

Given: $d/dt \; \underline{x}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t)$ (E-1)

Find: u(t) to minimize the quadratic performance index

$$PI = \int_0^T (\underline{x}^T \underline{Q}\,\underline{x} + \underline{u}^T \underline{P}\,\underline{u})\, dt \tag{E-2}$$

where $\underline{Q}$ is positive definite and $\underline{P}$ is at least positive semi-definite. Both $\underline{Q}$ and $\underline{P}$ are symmetric.

Using calculus of variations, the solution is found to be

$$\underline{u}(\underline{x}, t) = -\underline{K}(t)\,\underline{x}(t) \tag{E-3}$$

where

$$K(t) = \underline{P}^{-1}\,\underline{B}^T\,\underline{R}(t) \tag{E-4}$$

and $\underline{R}(t)$ is the solution to the nonlinear differential equation

$$\dot{\underline{R}}(t) + \underline{Q} - \underline{R}(t)\underline{B}\,\underline{P}^{-1}\underline{B}^T\,\underline{R}(t) + \underline{R}(t)\,\underline{A} + \underline{A}^T\underline{R}(t) = \underline{O} \tag{E-5}$$

subject to the boundary condition

$$\underline{R}(T) = \underline{\text{constant}} \tag{E-6}$$

Equation (E-5) is the matrix Riccati equation. Reference (5) is one of many references giving excellent detailed descriptions of the derivation of the Riccati equation. Readers so inclined are referred to those references.

The requirement of this thesis is the solution of the Riccati equation as t goes to infinity. The principal techniques for the solution follow.

2. Integration Method

The most obvious method of solving the matrix Riccati equation is by integration of the n simultaneous nonlinear differential equations from the boundary condition to a steady state value.

For the controller problem, the integration is performed backwards from the boundary condition, $\underline{R}(\infty) = \underline{O}$. Whereas for the filter problem, the integration is performed forward in time from $\underline{R}(O) = E\left[\underline{x}(0)\,\underline{x}^T(0)\right]$.

The subroutine named MRS (Appendix F) is capable of this integration.

3. Eigenvector Decomposition

This method is commonly referred to as Potter's method. It yields the steady state solution of the matrix Riccati equation (reduced Riccati equation), provided the coefficient matrices are constant, ie.,

$$\underline{Q} + \underline{R}\,\underline{A} + \underline{A}^T\underline{R} - \underline{R}\,\underline{B}\,\underline{P}^{-1}\underline{B}^T\underline{R} = \underline{O} \qquad \text{(E-7)}$$

The solution is found in terms of the eigenvectors of the 2n X 2n partitioned matrix

$$\underline{M} = \left[\begin{array}{c|c} \underline{A}^T & \underline{Q} \\ \hline \underline{B}\,\underline{P}^{-1}\underline{B}^T & -\underline{A} \end{array}\right] \qquad \text{(E-8)}$$

The eigenvalues and corresponding eigenvectors of $\underline{M}$ are found:

$$\underline{\lambda} = \left[\begin{array}{c|c} \underline{\lambda}_+ & \underline{O} \\ \hline \underline{O} & \underline{\lambda}_- \end{array}\right] \qquad \text{(E-9)}$$

where the subscripts denote the signs of the real parts of the eigenvalues, and the corresponding eigenvectors are

$$\underline{V}' = \left[\begin{array}{c|c} \underline{V}_{T+} & \underline{V}_{T-} \\ \hline \underline{V}_{B+} & \underline{V}_{B-} \end{array}\right] \tag{E-10}$$

$\underline{V}'$ is partitioned and the solution to the problem is

$$\underline{R} = \underline{V}_{T+} \quad \underline{V}_{B+}^{-1} \tag{E-11}$$

Reference (5) contains the proof of the method. The eigenvalue/ eigenvector decomposition was performed using EISPAC (Appendix F) and these results were manipulated by the subroutine named MRA (Appendix F).

4. Controller/Filter Duality

As has been previously implied, the optimal controller and Kalman filter problems are dual problems. Table e-1 illustrates the duality.

Aspect	Optimal Controller	Kalman Filter
State Equation	$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u}$ $(\underline{u} = -\underline{K}\ \underline{x})$	$x = \underline{A}\,\underline{x} + \underline{K}\,(\underline{z} - \underline{y}) + \underline{B}\,\underline{u}$
Performance Index	$\int_o^T (\underline{x}^T \underline{Q}\,\underline{x} + \underline{u}^T \underline{P}\,\underline{u})dt$	$\int_o^T (\underline{x}^T \underline{B}\,\underline{Q}\,\underline{B}^T \underline{x} + \underline{u}^T \underline{P}\,\underline{u})\,dt$
Matrix Riccati Equation	$-\dot{\underline{R}} = \underline{Q} - \underline{R}\,\underline{B}\,\underline{P}^{-1}\underline{B}\,\underline{R}$ $+ \underline{R}\,\underline{A} + \underline{A}^T \underline{R}$	$\dot{\underline{R}} = \underline{B}\,\underline{Q}\,\underline{B}^T - \underline{R}\,\underline{C}^T \underline{P}^{-1} \underline{C}\,\underline{R}$ $+ \underline{R}\,\underline{A}^T + \underline{A}\,\underline{R}$
Integration Technique	$\Delta t < 0\ ,\ \underline{R}(\infty) = \underline{O}$	$t > 0\ ,\ \underline{R}(0) = E\left[\underline{x}(0)\,x^T(0)\right]$
Potter's Method	$\underline{M} = \begin{bmatrix} \underline{A}^T & \underline{Q} \\ \underline{B}\,\underline{P}^{-1}\underline{B}^T & -\underline{A} \end{bmatrix}$	$\underline{M} = \begin{bmatrix} \underline{A} & \underline{B}\,\underline{Q}\,\underline{B}^T \\ \underline{C}^T\underline{P}^{-1}\underline{C} & -\underline{A}^T \end{bmatrix}$
Gains	$\underline{K} = \underline{P}^{-1}\underline{B}^T \underline{R}$	$\underline{K} = \underline{R}\,\underline{C}^T \underline{P}^{-1}$

Table e-1. Optimal Controller/Kalman Filter Duality

Appendix F

Computer Programs

A listing of some of the computer programs used in this thesis appear herein. The programming language was FORTRAN and comments are interspersed for the reader's benefit. Due to the magnitude of the packages, ACCESS II and EISPAC are not listed.

ACCESS II is a collection of engineering application programs in the MIT Mechanical Engineering/Civil Engineering joint Computer Facility. It is an updated version of an earlier package, and is primarily the work of Professor Richard S. Sidell, Instructor in Mechanical Engineering. ACCESS II is run on an INTERDATA M70 computer. The following portions of the package were used for this thesis:

CONTROL	Forms a partitioned matrix like that on page 22 and determines the rank of the matrix to establish system controllability or observability,
EIGENVALUES	Computes the eigenvalues of a system matrix,
MRA	Computes the steady state solution to the matrix Riccati equation using Potter's method, and
EISPAC	Calculates complex eigenvalues and eigenvectors of a general real matrix. It was recently incorporated with ACCESS II as a subroutine for MRA. EISPAC is taken from EISPACK, an eigensystem problem solver package developed through a National Science Foundation project known as NATS (National Activity to Test Software). The subroutines were tested principally at Argonne National Laboratory, the University of Texas, and Stanford University.

The program and associated subroutines for the computer simulations is listed herein. Subroutines called but not listed are described in reference (10) and are:

MINV	Inverts a square matrix, and
GAUSS	Generates a random, normally distributed variable

RKGS	Uses the Runge-Kutta method to obtain an approximate solution to a set of first order ordinary differential equations.

Two other subroutines, MRS and MRDOT, are listed. These were taken from ACCESS II and modified for the IBM 360 used by the Author. These subroutines calculate the steady state solution of the matrix Riccati equation using the integration technique.

MAIN

```
THIS IS THE MAIN PROGRAM WHICH IS USED TO SET UP THE
MATRICES AND INITIAL CONDITIONS FOR THE COMPUTER SIMULATIONS
      DIMENSION VEC1(9),VEC2(9),V(9,9),W(9,9),ZZ(9,3),ZCO(14),
     1 XCO(8),ADWI(6),XGWT(6),X(15),DERX(15),PRMT( 5),AUX(8,15)
      COMMON DISTYX(3),A(9,9),B(9,2),BFOR(9,3),K(6,6),DEFMIN(2),
     1 DEFMAX(2),DRMIN(2),DRMAX(2),KI,KO,CGAIN(2,6),BUBBLE,
     2         IPRINT,U(2),FOR(3),FORSYS(3),DTIMIV(2),IDID,
     3 AF(6,6),PF(6,5),UF(5),       CORREC(6),ESER(6),TPRINT
     4,ITSTEP,XORD,IX,NOISE(7),STNDF(3),STNDM(7),XMSR(7)
     5,MOG,XGF,XGO,FLDRAT,FLDR,TSTART,TSTOP,TVEC(30),XVEC(30),
     6 WVEC(30)
      REAL MASSP,MOMINR,LN,MOG,MO,MBOW,MDQ,MDW,MQ,MQN,MS,
     1 MSN,MSTAR,MW,M1W1,MWN,M1W1N,MWW,MW1W1,MW1W1N,MDELDW,
     2 MDELW,MINRP,MCO(15),K,MGE,NOISE
      EXTERNAL FCT,OUTP,LESQR
      KI=5
      KO=6
      GRAV=32.1725
      RAD=57.295779513
      DO 2010 I=1,9
      DO 2000 J=1,9
      V(I,J)=0.
      W(I,J)=0.
2000  CONTINUE
      DO 2005 J=1,3
2005  ZZ(I,J)=0.0
      DO 2010 J=1,2
      B(I,J)=0.0
2010  CONTINUE
      B(5,1)=1.0
      B(6,2)=1.0
      V(1,1)=1.
      V(3,3)=1.
      V(5,5)=1.
      V(6,6)=1.
      V(7,7)=1.0
      V(8,8)=1.0
      V(9,9)=1.0
      W(1,2)=1.
      W(3,4)=1.
      ZZ(2,1)=1.0
      ZZ(4,2)=1.
      ZZ(7,3)=1.0
READ IDENTITY AND NONDIMENSIONALIZED COEFFICIENTS
      READ(KI,1000) ID,LN,MASSP,MINRP,XBP,XGP,ZBP,ZGP
      READ(KI,1007) (ZCO(I),I=1,14)
      READ(KI,1003) (MCO(I),I=1,15)
      READ(KI,1003) (XCO(I),I=1,8)
READ MEAN DISTURBANCES AND STANDARD DEVIATIONS OF SYSTEM
DISTURBANCES AND MEASUREMENT NOISES
```

MAIN

```
      READ(KI,1003) (DISTMN(I),I=1,3)
      READ(KI,1003) (STNDF(I),I=1,3)
      READ(KI,1003) (STNDM(I),I=1,3)
   18 READ(KI,1003) UC,BUBBLE,DENS
      READ(KI,1100) NADMAS
READ MAGNITUDE(LBS) AND LCG(FT) OF ADDED WEIGHTS
      READ(KI,1002) (ADWT(I),XGWT(I),I=1,NADMAS)
READ CONTROLLER GAINS
      READ(KI,1006) ((CGAIN(I,J),J=1,6),I=1,2)
READ FILTER GAINS
      READ(KI,1004) ((K(I,J),J=1,4),I=1,6)
READ CONSTRAINTS ON PLANES
      READ(KI,1004) (DEFMIN(I),DEFMAX(I),I=1,2)
      READ(KI,1004) (DEMIN(I),DEMAX(I),I=1,2)
      WRITE(KO,3214) ID,UC,BUBBLE,DENS
      WRITE(KO,3215) (DISTMN(I),I=1,3)
CONVERT VELOCITIES FROM KNOTS TO FT/SEC
      UC=UC*1.688944
      THETA=BUBBLE/RAD
      TANT=TAN(THETA)
      TANT2=TANT*TANT
      TANT3=TANT2*TANT
      SINT=SIN(THETA)
      COST=COS(THETA)
      COST2=COST*COST
      C2=0.5*DENS*LN*LN
      C3=0.5*DENS*LN**3.0
      C4=0.5*DENS*LN**4.0
      C5=0.5*DENS*LN**5.0
ESTIMATE UO BASED UPON UC AND BUBBLE
      AA=XCO(5)+XCO(1)+TANT2*(ZCO(9)+TANT*ZCO(12)+XCO(7))+
     1 ABS(TANT)*(TANT*ZCO(10)+TANT2*ZCO(13))+TANT*ZCO(8)
      BB=UC*XCO(2)
      CC=UC*UC*XCO(3)+TANT*DISTMN(1)/C2
      ROOT=BB*BB-4.0*AA*CC
      UO=(-BB-SQRT(ROOT))/(2.0*AA)
      WO=UO*TANT
      ABSWO=ABS(WO)
      VEL=UO/COST
      VSQ=VEL*VEL
      WRITE(KO,3724) UC,UO,VEL,WO
      CN2=-UC*COST/UO/UO
      CN1=-CN2*UO-1.0
DIMENSIONALIZE COEFFICIENTS
      BO=C3*MASSP*GRAV
      DISP=BO/2240.
      MOMINR=C5*MINRP
      XB=XBP*LN
      XG=XGP*LN
      ZB=ZBP*LN
```

```
      ZG=ZGP*LN
      ZBOW=C2*VSQ*ZCO(1)
      ZDQ=C4*ZCO(2)
      ZDW=C3*ZCO(3)
      ZQ=C3*VEL*ZCO(4)
      ZQN=ZCO(5)*C3*VEL
      ZS=C2*VSQ*ZCO(6)
      ZSN=C2*VSQ*ZCO(7)
      ZSTAR=C2*VSQ*ZCO(8)
      ZW=C2*VEL*ZCO(9)
      Z1W1=C2*VEL*ZCO(10)
      ZWN=C2*VEL*ZCO(11)
      ZWW=C2*ZCO(12)
      ZW1W1=C2*ZCO(13)
      ZW1W1N=C2*ZCO(14)
      MBOW=C3*VSQ*MCO(1)
      MDQ=C5*MCO(2)
      MDW=C4*MCO(3)
      MQ=C4*VEL*MCO(4)
      MQN=C4*VEL*MCO(5)
      MS=C3*VSQ*MCO(6)
      MSN=C3*VSQ*MCO(7)
      MSTAR=C3*VSQ*MCO(8)
      MW=C3*VEL*MCO(9)
      M1W1=C3*VEL*MCO(10)
      MWN=C3*VEL*MCO(11)
      M1W1Q=C4*MCO(12)
      MWW=C3*MCO(13)
      MW1W1=C3*MCO(14)
      MW1W1N=C3*MCO(15)
      AI=C2*XCO(1)
      BI=C2*XCO(2)
      CI=C2*XCO(3)
      XDU=C3*XCO(4)
      XUU=C2*XCO(5)
      XWQ=C3*XCO(6)
      XWW=C2*XCO(7)
      XWWN=C2*XCO(8)
ALCULATE THE PROPER TRIM
      MOG=BO-Z1W1*ABSWO-COST*ZSTAR-WO*(ABSWO*(ZW1W1+CN1*
     1ZW1W1N)+WO*ZWW+COST*(ZW+CN1*ZWN))/COST-DISTMN(1)/COST
      WTMBO=MOG-BO
      MO=MOG/GRAV
      XGOMOG=XB*BO+M1W1*ABSWO+COST*MSTAR-TANT*(ZG*MOG-ZB*BO)+
     1WO*(ABSWO*(MW1W1+CN1*MW1W1N)+COST*(MW+CN1*MWN)+WO*MWW)/
     2COST+DISTMN(2)/COST
      XGOMO=XGOMOG/GRAV
      XGO=XGOMO/MO
      WRITE(KO,3371)DISP,WTMBO,XGO
ALCULATE ERRORS IN TRIM
```

MAIN

```
      XGMG=XGOMOG
      MGE=0.0
      DO 41 I=1,NADMAS
      MGE=MGE+ADWT(I)
   41 XGMG=XGMG+XGWT(I)*ADWT(I)
      WGHT=MOG+MGE
      XGACT=XGMG/WGHT
      XGE=XGACT-XGO
      WRITE(KO,3372)MGE,XGE
C CALCULATE STEADY STATE FORCES
      FORSYS(1)=COST*(MOG-BO+Z1W1*ABSWO+COST*ZSTAR)+WO*(ABSWO*
     1 (ZW1W1+CN1*ZW1W1N)+WO+ZWW+COST*(ZW+CN1*ZWN))
      FORSYS(2)=-SINT*(ZG*MOG-ZB*BO)+WO*(ABSWO*(MW1W1+CN1*MW1W1N)
     2+COST*(MW+CN1*MWN)+WO*MWW)+COST*(XB*BO-XGOMOG+M1W1*
     3 ABSWO+COST*MSTAR)
      FORSYS(3)=WO*WO*(XWW+CN1*XWWN)+UO*(UO*(AI+XUU)+BI*UC)+
     1 CI*UC*UC-SINT*(MOG-BO)
C CALCULATE ELEMENTS OF THE MATRICES
      ZDELW=2.0*ABSWO*(ZW1W1+CN1*ZW1W1N)+2.0*ZWW*WO+
     1COST*(ZW+CN1*ZWN)
      ZDELDW=MO-ZDW
      MDELW=2.0*ABSWO*(MW1W1+CN1*MW1W1N)+2.0*MWW*WO+
     1COST*(MW+CN1*MWN)
      MDELDW=-XGOMO-MDW
      XDELW=2.0*WO*(XWW+CN1*XWWN)
      V(2,2)=ZDELDW/COST
      V(2,4)=-XGOMO-ZDQ
      V(2,7)=ZDELDW*TANT
      V(2,9)=WO/GRAV
      V(4,2)=MDELDW/COST
      V(4,4)=MOMINR-MDQ
      V(4,7)=ZG*MO+MDELDW*TANT
      V(4,9)=(ZG*UO-XGO*WO)/GRAV
      V(7,4)=MO*ZG
      V(7,7)=MO-XDU
      V(7,9)=UO/GRAV
      W(2,2)=ZDELW/COST
      W(2,3)=ZDELW*UO
      W(2,4)=COST*(ZQ+CN1*ZQN)+UO*(MO-ZDELDW*(1.0+TANT2))
      W(2,5)=COST2*ZBOW
      W(2,6)=COST2*(ZS+CN1*ZSN)
      W(2,7)=WO*CN2*(ZW1W1N*ABSWO+COST*ZWN)+ZDELW*TANT
      W(2,9)=COST
      W(4,2)=MDELW/COST
      W(4,3)=UO*MDELW-COST*(ZG*MOG-ZB*BO)
      W(4,4)=MO*(-XGO*UO-ZG*WO)-UO*MDELDW*(1.0+TANT2)+
     1M1W1Q*ABSWO+COST*(MQ+CN1*MQN)
      W(4,5)=MBOW*COST2
      W(4,6)=COST2*(MS+CN1*MSN)
      W(4,7)=TANT*MDELW+WO*CN2*(MW1W1N*ABSWO+MWN*COST)
```

```
      W(4,8)=-MOG*COST
      W(4,9)=-XGO*COST-ZG*SINT
      W(7,2)=XDELW/COST
      W(7,3)=UO*XDELW-(MOG-BO)*COST
      W(7,4)=WO*(XWQ-MO)
      W(7,7)=2.0*UO*(AI+XUU)+BI*UC+CN2*XWWN*WO+WO+XDELW*TANT
      W(7,9)=-SINT
ODIFY MATRICES FOR ANGLES IN DEGREES
      DO 377 I=3,6
      V(2,I)=V(2,I)/. .
      W(2,I)=W(2,I)/RAD
      V(4,I)=V(4,I)/RAD
      W(4,I)=W(4,I)/RAD
      V(7,I)=V(7,I)/RAD
77    W(7,I)=W(7,I)/RAD
      WRITE(KO,3098)
      WRITE(KO,3110)((V(I,J),J=1,9),I=1,9)
      WRITE(KO,3099)
      WRITE(KO,3110)((W(I,J),J=1,9),I=1,9)
      CALL MINV(V,9,DETERM,VEC1,VEC2)
      CALL MPRD(V,W,A,9,9,9)
ODIFY A MATRIX FOR MGE IN MEGAPOUNDS
      DO 54 I=1,9
54    A(I,9)=A(I,9)*1000000.
ONVERT TRIM ERROR TO MEGAPOUNDS
      MGE=MGE/1000000.
      WRITE(KO,3022)
      WRITE(KO,3110)((A(I,J),J=1,9),I=1,9)
      CALL MPRD(V,ZZ,BFOR,9,9,3)
      WRITE(KO,3026)
      WRITE(KO,3130)((BFOR(I,J),J=1,3),I=1,9)
      WRITE(KO,3102)(FORSYS(I),I=1,3)
      WRITE(KO,2222)
      WRITE(KO,2223)((CGAIN(I,J),J=1,6),I=1,2)
      WRITE(KO,2224)
      WRITE(KO,2223)((K(I,J),J=1,4),I=1,6)
RGANIZE OBSERVER A & B MATRIX & INITIAL OF .
      DO 2304 I=1,4
      DO 2301 J=1,4
01    AF(I,J)=A(I,J)
      DO 2302 J=5,6
      JM4=J-4
      BF(I,JM4)=A(I,J)
      JP3=J+3
02    AF(I,J)=A(I,JP3)
      DO 2303 J=1,2
      JP2=J+2
03    BF(I,JP2)=BFOR(I,J)
04    BF(I,5)=A(I,7)
      DO 2306 I=5,6
```

```
    DO 2305 J=1,6
05  AF(I,J)=0.0
    DO 2306 J=1,5
06  BF(I,J)=0.0
    WRITE(KO,3023)
    WRITE(KO,2225) ((AF(I,J),J=1,6),I=1,6)
    WRITE(KO,3100)
    WRITE(KO,2227) ((BF(I,J),J=1,5),I=1,6)
ET UP FOR THE ORD. DIFF. EQN. SOLN.
    DO 315 I=1,3
    DTIMEV(I)=0.0
15  FOR(I)=FORSYS(I)+DISTMN(I)+DTIMEV(I)
    READ(KI,1008) (X(I),I=1,15),XORD
    U(1)=0.0
    U(2)=0.0
    UF(1)=X(5)
    UF(2)=X(6)
    UF(3)=FORSYS(1)
    UF(4)=FORSYS(2)
    UF(5)=X(7)
    X(8)=XGE
    X(9)=MGE
ET UP VECTORS FOR LEAST SQUARE SMOOTHER
    DO 352 I=2,30
    XVEC(I)=X(14)
    WVEC(I)=X(15)
52  TVEC(I)=-62.+2.0*FLOAT(I)
ALCULATE THE ERRORS IN THE ESTIMATES
    ESER(1)=X(10)-X(1)
    ESER(2)=X(11)-X(2)
    ESER(3)=X(12)-X(3)
    ESER(4)=X(13)-X(4)
    ESER(5)=X(14)-X(8)
    ESER(6)=X(15)-X(9)
ALCULATE THE CORRECTIONS FOR THE FILTER EQUATIONS
    DO 45 I=1,6
    CORREC(I)=0.0
    DO 45 J=1,4
45  CORREC(I)=CORREC(I)-K(I,J)*ESER(J)
RMT(1)=INITIAL TIME
RMT(2)=FINAL TIME
RMT(3)=INITIAL TIME INCREMENT
RMT(4)=ERROR BOUND
    READ(KI,1500) (PRMT(I),I=1,4),ITSTEP
    WRITE(KO,2004) (STNDF(I),I=1,3)
    WRITE(KO,2006) (STNDM(I),I=1,7)
    WRITE(KO,2729) (PRMT(I),I=2,4)
NITIALLY DERX IS RELATIVE INPUT ERROR WEIGHTS
    READ(KI,1008) (DERX(I),I=1,15)
EAD FLOODING CASUALTY PARAMETERS
```

MAIN

```
      READ(KI,1004)FLDRAT,XGF,TSTART,TSTOP
      IX=3479
      FLDR=0.
      IPRINT=0
      TPRINT=0.0
      IDID=14
      CALL RKGS (PRMT,Y,DERY,15,IHLF,FCT,OUTP,AUX)
000   FORMAT(I10,7F10.3)
002   FORMAT(2F10.3)
003   FORMAT(3F10.3)
004   FORMAT(4F10.3)
006   FORMAT(6F10.3)
007   FORMAT(7F10.3)
008   FORMAT(8F10.3)
100   FORMAT(I10)
500   FORMAT(4F10.3,3I10)
004   FORMAT(1H0,'EXTERNAL DISTURBANCE STANDARD DEVIATIONS:',
     1 1P3E14.6)
006   FORMAT(1H0,'MSR. NOISE STND. DEVIATIONS:',1P7E14.6)
222   FORMAT(1H1,30X,'CONTROLLER GAINS')
223   FORMAT(1H ,2(1P6E15.6,/,1X))
224   FORMAT(1H0,24X,'  FILTER GAINS')
225   FORMAT(1H ,5(1P6E15.6,/,1X))
227   FORMAT(1H ,6(1P5E14.6,/,1X))
228   FORMAT(1H ,8(1P4E15.6,/,1X))
728   FORMAT(1H0,'FINAL TIME=',F6.0,' SEC, TIME STEP=',F6.4,
     1', UPPER ERROR BOUND=',F6.3)
022   FORMAT(1H0,30X,'A MATRIX (ANGLES IN DEGREES AND TRIM',
     1' ERROR IN MEGAPOUNDS AND FEET)')
023   FORMAT(1H0,30X,'  FILTER A MATRIX')
026   FORMAT(1H ,10X,'BFOR MATRIX')
098   FORMAT(1H0,45X,'V MATRIX (ANGLES IN DEGREES)')
099   FORMAT(1H0,45X,'W MATRIX (ANGLES IN DEGREES)')
100   FORMAT(1H0,25X,'  FILTER B MATRIX')
102   FORMAT(1H ,'VEHICLE STEADY STATE FORCES:',1PE14.6,
     1' LBS,',E14.6,' FT-LBS,',E14.6,' LBS')
110   FORMAT(1H ,9(1P9E14.6,/,1X))
130   FORMAT(1H ,9(1P3E14.6,/,1X))
214   FORMAT(1H0,'ID=',I3,', TURNS FOR',F5.1,' KNOTS,',
     1' ORDERED BUBBLE=',F4.1,' DEGREES, DENSITY=',F7.4)
215   FORMAT(1H0,'MEAN DISTURBANCES ARE:',1P3E15.6)
371   FORMAT(1H0,'DISPLACEMENT=',1PE13.6,' TONS, WEIGHT MINUS',
     1' DISPLACEMENT=',E13.6,' LBS, XGO=',E13.6,3H FT)
372   FORMAT(1H0,'TRIM ERROR=',1PE13.6,' LBS, ',E13.6,' FT. IN LCG')
724   FORMAT(1H0,'VELOCITIES(FT/SEC): UC=',1PE13.6,5H, UC=
     1,E13.6,6H, VLL=,E13.6,5H, WO=,E13.6)
      STOP
      END
```

MAIN

```
      SUBROUTINE MPRD(A,B,C,NR,NK,NC)
THIS SUBROUTINE MULTIPLIES CONFORMAL MATRICES
      REAL A(1),B(1),C(1)
      DO 20 I=1,NR
      DO 20 J=1,NC
      T=0.0D0
      DO 10 K=1,NK
      IK=I+NR*(K-1)
      KJ=K+NK*(J-1)
0     T=T+A(IK)*B(KJ)
      IJ=I+NR*(J-1)
0     C(IJ)=T
      RETURN
      END
```

LESQR

```
      SUBROUTINE LESQR(T,XGEEST,WEEST,XGES,WES)
HE PURPOSE OF THIS SUBROUNTINE IS TO SMOOTH 30 SAMPLES OF
GE AND WE AND GIVE SMOOTHED ESTIMATES OF THE LATEST DATA
ARAMETERS ARE AS FOLLOWS:
NPUTS:
=TIME
GEEST=XG ERROR ESTIMATE
EEST=WEIGHT ERROR ESTIMATE
UTPUTS:
GES=SMOOTHED XG ERROR ESTIMATE
ES=SMOOTHED WEIGHT ERROR ESTIMATE
      COMMON DISTMN(3),A(9,9),B(9,2),BFOR(9,3),K(6,6),DEFMIN(2),
     1 DEFMAX(2),DTMIN(2),DRMAX(2),KI,KO,CGAIN(2,6),BUBBLE,
     2       IPRINT,U(2),FOR(3),FORSYS(3),DTIMEV(3),IDID,
     3 AF(6,6),BF(6,5),UF(5),       CORREC(6),ESER(6),TPRINT
     4,ITSTEP,XOPD,IX,NOISE(7),STNDF(3),STNDM(7),XMSP(7)
     5,MCG,XGF,XGO,FLDRAT,FLDR,TSTART,TSTOP,TVEC(30),XVEC(30),
     6 WVEC(30)
      REAL MX,MW
PDATE VECTORS CONTAINING DATA TO BE SMOOTHED
      DO 10 I=1,29
      IP1=I+1
      XVEC(I)=XVEC(IP1)
      WVEC(I)=WVEC(IP1)
10    TVEC(I)=TVEC(IP1)
      XVEC(30)=XGEEST
      WVEC(30)=WEEST
      TVEC(30)=T
      SUMT=0.0
      SUMX=0.0
      SUMW=0.0
      SUMT2=0.0
      SUMX2=0.0
      SUMW2=0.0
      SUMTX=0.0
      SUMTW=0.0
      DO 20 N=1,30
      SUMT=SUMT+TVEC(N)
      SUMX=SUMX+XVEC(N)
      SUMW=SUMW+WVEC(N)
      SUMT2=SUMT2+TVEC(N)**2
      SUMX2=SUMX2+XVEC(N)**2
      SUMW2=SUMW2+WVEC(N)**2
      SUMTX=SUMTX+TVEC(N)*XVEC(N)
20    SUMTW=SUMTW+TVEC(N)*WVEC(N)
      TM=SUMT/30.
      XM=SUMX/30.
      WM=SUMW/30.
      SIGT2=SUMT2-30.*TM**2
      SIGX2=SUMX2-30.*XM**2
```

LESQR

```
SIGW2=SUMW2-30.*WM**2
SIGTX=SUMTX-30.*TM*XM
SIGTW=SUMTW-30.*TM*WM
MX=SIGTX/SIGT2
MW=SIGTW/SIGT2
XGES=MX*(T-TM)+XM
WES=MW*(T-TM)+WM
RETURN
END
```

MAIN

```
      SUBROUTINE OUTP(T,X,DERX,IHLF,NDIM,PRMT)
C THIS SUBROUTINE IS CALLED BETWEEN TIME STEPS OF THE
C DIFFERENTIAL EQUATIONS SOLUTION
      DIMENSION X(15),DERX(15),PRMT( 5)
      COMMON DISTMN(3),A(9,9),B(9,2),BFOR(9,3),K(6,6),DEFMIN(2),
     1 DEFMAX(2),DRMIN(2),DRMAX(2),KI,KO,CGAIN(2,6),BUBBLE,
     2         IPRINT,U(2),FOR(3),FORSYS(3),DTIMEV(3),IDID,
     3 AF(6,6),BF(6,5),UF(5),       CORREC(6),ESER(6),IPRINT
     4,ITSTEP,XORD,IY,NOISE(7),SINDF(3),SINDM(7),XMSR(7)
     5,MOG,XGF,YGO,FLDRAT,FLDR,TSTART,TSTOP,TVEC(30),XVEC(30),
     6 WVEC(30)
      REAL K,NOISE,MOG
C
C ELEMENTS OF THE X VECTOR ARE AS FOLLOWS:
C X(1)=DEPTH
C X(2)=RATE OF DESCENT
C X(3)=PITCH ERROR
C X(4)=RATE OF PITCH
C X(5)=FAIRWATER PLANE DEFLECTION
C X(6)=STERN PLANE DEFLECTION
C X(7)=FORWARD VELOCITY DEVIATION
C X(8)=XG ERROR
C X(9)=WEIGHT ERROR
C X(10)=DEPTH ESTIMATE
C X(11)=RATE OF DESCENT ESTIMATE
C X(12)=PITCH ERROR ESTIMATE
C X(13)=PITCH RATE ESTIMATE
C X(14)=XG ERROR ESTIMATE
C X(15)=WEIGHT ERROR ESTIMATE
C
      FLDR=FLDRAT
      IF(T.LT.TSTART) FLDR=0.
      IF(T.GE.TSTOP) FLDR=0.
C GENERATE SYSTEM AND MEASUREMENT NOISE
      DO 529 I=1,3
      CALL GAUSS(IX,SINDF(I),0.0,DTIMEV(I))
  529 FOR(I)=FORSYS(I)+DISTMN(I)+DTIMEV(I)
      DO 534 I=1,7
      CALL GAUSS(IX,SINDM(I),0.0,NOISE(I))
  534 XMSR(I)=X(I)+NOISE(I)
C DEPTH CHANGE ORDERS
      IF(T.GT.30.)XORD=180.
      IF(T.GT.60.)XORD=210.
      IF(T.GT.120.)XORD=180.
      IF(T.GT.200.)XORD=210.
      IF(T.GT.280.)XORD=180.
C CALCULATION OF OPTIMAL CONTROL
      DO 70 I=1,2
      U(I)=CGAIN(I,1)*(XMSR(1)-XORD)
```

OUTP

```
      DO 70 J=2,6
   70 U(I)=U(I)+CGAIN(I,J)*XMSR(J)
IMPOSITION OF CONSTRAINTS ON PLANES
      DO 50 I=1,2
      IDEF=4+I
      IF(U(I))10,50,30
   10 CONTINUE
      IF(X(IDEF).LE.DEFMIN(I))GO TO 47
      IF(U(I).GE.DRMIN(I))GO TO 50
      U(I)=DRMIN(I)
      GO TO 50
   30 CONTINUE
      IF(X(IDEF).GE.DEFMAX(I))GO TO 48
      IF(U(I).LE.DRMAX(I))GO TO 50
      U(I)=DRMAX(I)
      GO TO 50
   47 X(IDEF)=DEFMIN(I)
      GO TO 49
   48 X(IDEF)=DEFMAX(I)
   49 U(I)=0.
   50 CONTINUE
MEASURE UF FOR THE FILTER
      UF(1)=XMSR(5)
      UF(2)=XMSR(6)
      UF(5)=XMSR(7)
CALCULATE THE APPARENT & REAL ERRORS IN THE ESTIMATES
      ESER(1)=X(10)-XMSR(1)
      ESER(2)=X(11)-XMSR(2)
      ESER(3)=X(12)-XMSR(3)
      ESER(4)=X(13)-XMSR(4)
      ESER(5)=X(14)-X(8)
      ESER(6)=X(15)-X(9)
CALCULATE THE CORRECTIONS FOR THE FILTER EQUATIONS
      DO 45 I=1,6
      COFREC(I)=0.0
      DO 45 J=1,4
   45 COFREC(I)=COFREC(I)-K(I,J)*ESER(J)
CONTROL PRINTING OF THE OUTPUT
      IF(T.LT.TPRINT)GO TO 3
      IPRINT=IPRINT+ITSTEP
      ITEMP=IHLF+1
      TPRINT=FLOAT(IPRINT)-PRMT(3)/FLOAT(ITEMP)
SMOOTH ESTIMATE OF STATE OF TRIM
      CALL LESQR(T,X(14),X(15),XGES,WES)
      IF(IDID.LT.11)GO TO 8
      WRITE(KO,2001)
      IDID=0
    8 XBUB=X(3)+BUBBLE
      WRITE(KO,3000)T,U(1),U(2),X(1),X(2),XBUB,X(4),
     1 X(8),X(9),IHLF
```

OUTP

```
      WRITE(KO,3001)X(7),X(5),X(6),(ESER(I),I=1,6)
      WRITE(KO,3001)X(14),X(15),XORD,(CORREC(I),I=1,6)
      WRITE(KO,3001)XMSR(7),XMSR(5),XMSR(6),(XMSR(I),I=1,4)
     1,XGES,WES
      IDID=IDID+1
    2 CONTINUE
 1001 FORMAT(1H1,T6,'TIME',T19,'BP RATE',T33,'SP RATE',
     1 T49,'DEPTH'        ,T59,'DESCENT RATE',T76,'PITCH',T88,
     2'PITCH RATE',T102,'XG ERROR',T116,'MG ERROR',T128,
     3'IHLF',/,T5,'DELTA U',T19,'BP DEF.',T33,'SP DEF.',
     4 4(7X,'EST-MSR'),2(7X,'EST-ACT'),/,T5,
     5'XGE EST',T19,'MGE EST',T30,'ORDERED DEPTH',T48,
     6'CORREC',5(8X,'CORREC'),/,T1,7(3X,'MEASUREMENT'),
     7 2(2X,'SMOOTHED EST'))
 3000 FORMAT(1H0,F9.3,' SEC.',1P8E14.6,I3)
 3001 FORMAT(1H ,1P9E14.6)
      RETURN
      END
```

MAIN

```
      SUBROUTINE FCT(T,X,DERX)
 HIS SUBROUTINE CALCULATES THE RIGHT HAND SIDES OF THE
 IFFERENTIAL EQUATIONS FOR THE RUNGE-KUTTA SOLUTION
      DIMENSION X(15),DERX(15)
      COMMON DISTMN(3),A(9,9),B(9,2),BFOR(9,3),K(6,6),DEFMIN(2),
     1 DEFMAX(2),DRMIN(2),DRMAX(2),KI,KO,CGAIN(2,6),BUBBLE,
     2       IPRINT,U(2),FOR(3),FORSYS(3),DTIMEV(3),IDID,
     3 AF(6,6),BF(6,5),UF(5),       CORREC(6),ESER(6),TPRINT
     4,ITSTEP,XORD,IX,NOISE(7),SINDF(3),SINDM(7),XMSR(7)
     5,MOG,XGF,XGO,FLDRAT,FLDR,ISTART,TSTOP,TVEC(30),XVEC(30),
     6 WVEC(30)
      REAL K,NOISE,MOG
 TATE EQUATIONS
      DO 30 I=1,9
      DERX(I)=0.
      DO 10 J=1,9
  10  DERX(I)=DERX(I)+A(I,J)*X(J)
      DO 20 J=1,2
  20  DERX(I)=DERX(I)+B(I,J)*U(J)
      DO 30 J=1,3
  30  DERX(I)=DERX(I)+BFOR(I,J)*FOR(J)
      DERX(8)=FLDR*(XGF-XGO-X(8))/(MOG+X(9)*1000000.)
      DERX(9)=FLDR/1000000.
 ILTER EQUATIONS
      DO 42 I=1,6
      IP9=I+9
      DERX(IP9)=CORREC(I)
      DO 41 J=1,6
      JP9=J+9
  41  DERX(IP9)=DERX(IP9)+AF(I,J)*X(JP9)
      DO 42 J=1,5
  42  DERX(IP9)=DERX(IP9)+BF(I,J)*UF(J)
      RETURN
      END
```

MAIN

```
      SUBROUTINE MRS(A,B,QO,PO,RO,R,KT,DT,NROW,NCX,IPNT)
THIS IS A MODIFICATION OF THE SUBROUTINE FROM THE ACCESS II
PACKAGE AT MIT

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THE ARGUMENTS FOR THIS SUBROUTINE ARE AS FOLLOWS:
A AND B ARE COEFFICIENT MATRICES
QO IS THE Q MATRIX FROM THE PERFORMANCE INDEX
PO IS THE INVERSE OF P FROM THE PERFORMANCE INDEX
RO IS THE BOUNDARY CONDITION FOR THE RICCATI EQUATION
R IS THE SOLUTION OF THE MATRIX RICCATI EQUATION
KT IS THE GAINS MATRIX
DT IS THE TIME STEP
NROW IS THE DIMENSION OF THE STATE VECTOR
NCX IS THE NUMBER OF COLUMNS IN THE B MATRIX
IPNT IS THE PRINTER'S ADDRESS
       INPUT ARGUEMENTS AND RESULT
      REAL A(1),B(1),QO(1),PO(1),RO(1),R(1)
       TEMPORARY STORAGE ARRAYS
       BPB,RA,AND RG ARE GENERAL, ALL OTHERS ARE SYMMETRIC
      REAL BPB(100),RK(100),RD(100),RDT(100),RMAX(100),Q(100)
     1,BT(100),RA(100),RG(100),KT(NCX,NROW)
      NLOOPS=0
 10   H=DT/6.0
      NQ=NROW*(NROW+1)/2
      TIME SCALE EQUATION
      CALCULATE P=B*P*TRA(B)
      FIRST GET RK=P*TRA(B)
      DO 60 I=1,NCX
      DO 60 J=1,NROW
      T=0.0
      DO 50 K=1,NCX
      IA=I+(K-1)*NCX
      IAT=K+(I-1)*NCX
      IB=J+(K-1)*NROW
0     T=T+(PO(IA)+PO(IAT))*B(IB)
      IC=I+(J-1)*NCX
0     RK(IC)=0.5*T
      THEN GET P=B*RK
       EVEN THOUGH RESULT IS SYMMETRIC, DO FULL MATRIX FOR CONVENIENCE
      IC=0
      DO 80 J=1,NROW
      DO 80 I=1,NROW
      T=0.0
      DO 70 K=1,NCX
      IA=I+(K-1)*NROW
      IB=K+(J-1)*NCX
0     T=T+B(IA)*RK(IB)
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MRS

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      IC=IC+1
0     BPB(IC)=H*T
      CONVERT R AND Q TO SYMMETRIC FORM AND FINISH TIME SCALING
      IC=0
      DO 90 J=1,NROW
      DO 90 I=1,J
      IA=I+(J-1)*NROW
      IB=J+(I-1)*NROW
      IC=IC+1
      R(IC)=0.5*(RO(IA)+RO(IB))
0     Q(IC)=0.5*H*(QO(IA)+QO(IB))
      TIME=0.0
      WRITE(IPNT,5000)
000   FORMAT('0',T8,'T',T19,'DT',T28,'ERROR',T37,'CONTROLLER GAINS')
      DO 190 I=1,NQ
90    RMAX(I)=0.0
      RUNGE-KUTTA INTEGERATION
00    DO 250 K=1,10
      DO 210 I=1,NQ
10    BT(I)=R(I)
      CALL MRDOT(A,BPB,Q,BT,RA,RG,RDT,H,NROW)
      DO 220 I=1,NQ
      RD(I)=RDT(I)
20    BT(I)=R(I)+3.0*RD(I)
      CALL MRDOT(A,BPB,Q,BT,RA,RG,RDT,H,NROW)
      DO 230 I=1,NQ
      BT(I)=R(I)+3.0*RDT(I)
30    RD(I)=RD(I)+2.0*RDT(I)
      CALL MRDOT(A,BPB,Q,BT,RA,RG,RDT,H,NROW)
      DO 240 I=1,NQ
      BT(I)=R(I)+6.0*RDT(I)
40    RD(I)=RD(I)+2.0*RDT(I)
      CALL MRDOT(A,BPB,Q,BT,RA,RG,RDT,H,NROW)
      TIME STEP IS COMPLETE
      UPDATE R AND COMPUTE ERROR NORM
      DRS=0.0
      DO 250 I=1,NQ
      DR=RD(I)+RDT(I)
      R(I)=R(I)+DR
      DR=ABS(DR/DT)
      RMAX(I)=AMAX1(DR,RMAX(I))
      IF(RMAX(I).NE.0.0)DRS=AMAX1(DRS,DR/RMAX(I))
50    CONTINUE
      COMPUTE AND PRINT CONTROLLER GAINS
      K=P*TRA(B)*R
      TIME=TIME+10.0*DT
      DO 280 I=1,NCX
      DO 270 J=1,NROW
      T=0.0
      DO 260 K=1,NROW
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MRS

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    IA=I+(K-1)*NCX
    GET ELE(K,J) FROM A SYMMETRIC MATRIX
    IF(K.LE.J)IB=J*(J-1)/2+K
    IF(K.GT.J)IB=K*(K-1)/2+J
60  T=T+PK(IA)*R(IB)
    KT(I,J)=T
70  RD(J)=T
    IF(I.EQ.1)WRITE(IPNT,5001)TIME,DT,DRS,(RD(J),J=1,NROW)
001 FORMAT(1H , 1PG11.4,10G12.4,/(' ',40X,9G12.4))
    IF(I.GT.1)WRITE(IPNT,5002)(RD(J),J=1,NROW)
002 FORMAT(1H ,35X, 1P8G12.4,/(' ',40X,8G12.4))
80  CONTINUE
TERMINATE IF NOT CONVERGING
    IF(TIME.LT.-180.0)GO TO 572
     TERMINATE IF DRS IS SMALL
    IF(DRS.GT.0.001)GO TO 200
    GO TO 573
572 NLOOPS=NLOOPS+1
    IF(NLOOPS.GT.1)GO TO 573
    DT=DT/2.
    GO TO 10
573 RETURN
    END
```

MAIN

```
      SUBROUTINE MRDOT(A,BPB,Q,BT,RA,RG,RDT,H,NROW)

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THIS SUBROUTINE IS CALLED BY MRS
      IMPLICIT INTEGER*2(I-N)
      REAL A(1),BPB(1),Q(1),BT(1),RA(1),RG(1),RDT(1)
      RDOT(I,L)=-R(I,K)*A(K,L)-A(K,I)*R(K,L)-Q(I,L)+R(I,J)*P(J,K)*R(K,L)
      NOTE THAT B CONTAINS CURRENT VERSION OF R FOR RUNGE-KUTTA
      FOR R*A+TRA(A)*R USE FORMULA:
      TERM(I,L)=R(I,K)*A(K,L)+A(K,I)*R(K,L)
      FOR R*P*R USE FORMULA:
      TERM(I,L)=R(I,J)*P(J,K)*R(K,L)
-----COMPUTE R*A NAD R*TRA(B)*INV(P)*B=R*P SINCE P PRESET
      IJ=0
      DO 50 J=1,NROW
      DO 50 I=1,NROW
      TA=0.0
      TB=0.0
      IK=I*(I-1)/2
      KJ=(J-1)*NROW
      DO 40 K=1,NROW
      IF(K.LE.I)GO TO 20
      IK=IK+K-1
      GO TO 30
0     IK=IK+1
0     KJ=KJ+1
      TA=TA+BT(IK)*A(KJ)
0     TB=TB+BT(IK)*BPB(KJ)
      IJ=IJ+1
      RG(IJ)=TB
0     RA(IJ)=TA*H
----- NOW COMPUTE NEW RDOT
      IJ=0
      DO 100 J=1,NROW
      IL=(J-1)*NROW
      LI=J-NROW
      DO 100 I=1,J
-----COMPUTE RG*R
      TA=0.0
      KJ=J*(J-1)/2
      IK=I-NROW
      DO 90 K=1,NROW
      IF(K.LE.J)GO TO 70
      KJ=KJ+K-1
      GO TO 80
0     KJ=KJ+1
0     IK=IK+NROW
0     TA=TA+RG(IK)*BT(KJ)
```

MRDOT

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      IJ=IJ+1
      IL=IL+1
      LI=LI+NROW
)0    RDT(IJ)=TA-Q(IJ)-RA(IL)-RA(LI)
      RETURN
      END
```

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